The Laplacian operator

on graphs, simplicial complexes, and on multicomplexes

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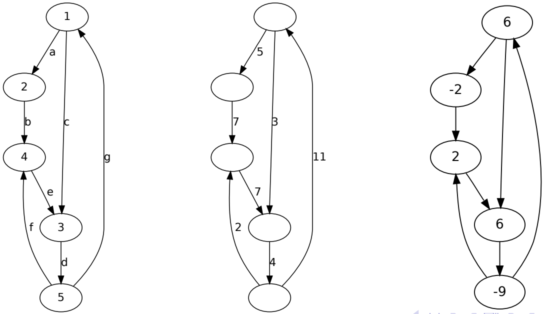
Graph Laplacian

Discrete analogs of div and grad

Let G be an oriented graph. The discrete analog of a vector field is a function defined on the *edges*, and the divergence operator maps such functions to functions on the *vertices* by

$$D(f)(v) = \sum_{a \xrightarrow{e} v} f(e) - \sum_{v \xrightarrow{e} b} f(e).$$



Graph Laplacian

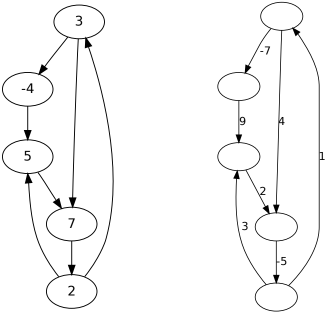
W.r.t. the natural basis, this operator has matrix

$$D = \begin{array}{ccccccc} & a & b & c & d & f & g \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & -1 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & -1 & -1 \end{array}$$

Graph Laplacian

The discrete analog of the gradient is the operator D^T which maps functions defined on vertices to functions defined on edges:

$$D^T(g)(e) = g(b) - g(a) \qquad a \xrightarrow{e} b$$



Graph Laplacian

W.r.t. the natural basis, this operator has matrix

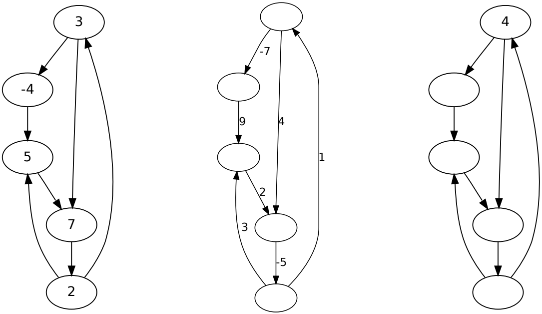
$$D^T = \begin{pmatrix} & a & b & c & d & f & g \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & -1 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}^T$$

Graph Laplacian

The discrete analog of the laplacian operator "div" compose "grad" is then

$$Q = DD^T$$

which is an operator that maps functions defined on vertices to functions defined on vertices.



The matrix of Q is

$$\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

Facts about graph Laplacian Q

Suppose G has n vertices, c components, $Q = DD^T$, Δ valency, A adjacency (of undirected graph, so symmetric).

- Symmetric, positive definite.
- Does *not* depend on the orientation! Depends on ordering of vertices, but only up to permutation similarity.
- Hence eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_1 \geq 0$.
- In fact, $\lambda_n = 0$ and $\lambda_1 \leq n$.
- $Q = \Delta - A$,
- $\mathrm{rk} Q = n - c$.
- $\lambda_{n-1} > 0$ iff $c = 1$.
- Eigenvalues of complement: $\lambda_i(\overline{G}) = n - \lambda_{n-i+2}$.
- If G k-regular then A has eigenvalues $\theta_i = k - \lambda_i$.
- As a quadratic form,

$$x^T Qx = \sum_{u \xrightarrow{e} v} (x_u - x_v)^2.$$

Spanning trees

Theorem

Let u be any vertex, and let $Q[u]$ denote the matrix obtained from Q by removing the u'th row and column. Then the number of spanning trees in G is

$$\det Q[u] = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}.$$

Majorization of the spectrum

- $S(G) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ non-increasing.
- $D(G) = (d_1, d_2, \dots, d_n)$ degree-sequence, non-increasing.

Theorem

$S(G)$ majorizes $D(G)$, i.e.,

$$\forall k : \sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} d_j.$$

In particular, $\lambda_1 \geq d_1$.

Majorization of the spectrum

Furthermore:

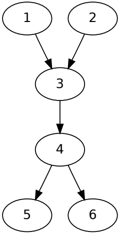
Theorem
$D(G)^T$ *(conjugate partition) majorizes* $D(G)$.

Conjecture
$D(G)^T$ *(conjugate partition) majorizes* $S(G)$.

Integrality of Laplacian spectra

Important question: when are all eigenvalues of Q integers?

- Iff same holds for complement $\overline{G}$.
- A tree has integral Laplacian spectrum iff it is a star, $G = K_{i,n-1}$.



$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$S(G) = (5/2 + 1/2\sqrt{17}, 3, 1, 1, 5/2 - 1/2\sqrt{17}, 0)$$

Simplicial complexes

- K simplicial complex.
- $C_i = C_i(K, \mathbb{R})$ chains.
- Differential

$$\partial_i : C_i \mapsto C_{i-1}$$
$$[v_0, \ldots, v_i] \mapsto \sum_j (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i]$$

- Homology $H_i = \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$
- Choose inner product, dual $\partial_i^* : C_i \rightarrow C_{i+1}$.
- Define Laplacians $L_i' = \partial_{i+1}\partial_{i+1}^*$, $L_i'' = \partial_i^* \partial_i$, $L_i = L_i' + L_i''$. (Duval and Reiner).
- L_i' direct generalization of graph Laplacian $Q = DD^T$. (Graphs are 1-dim s.c.)
- L_i'' direct generalization of edge Laplacian $D^T D$.
- L_i has useful connection to homology (Eckmann).

Spectra

Denote the spectra of L'_i, L''_i and L_i by s'_i, s''_i and s_i^{tot} (sorted multisets of nonneg real numbers). Use $\approx$ for equality up to the number of trailing zeroes.

- $s_i^{tot} \approx s'_i \cup s''_i \approx s'_i \cup s'_{i-1}$, in fact any of $(s_i)_{i\geq 0}$, $(s'_i)_{i\geq 0}$, or $(s''_i)_{i\geq 0}$ determine the other two.
- $$C_1 = \operatorname{im}\partial_{i+1} \oplus \ker L_i \oplus \operatorname{im}\partial i^*$$
  $$\operatorname{im}\partial_{i+1} \oplus \ker L_i = \ker \partial_i$$
  $$\ker L_i = H_i$$

  "Combinatorial Hodge theory"
- So, number of zero eigenvalues of L_i gives i'th Betti number. We will be interested in the non-zero eigenvalues.

Shifted simplicial complexes

Ground set of K is now $[n] = \{1, 2, \ldots, n\}$, with natural total order.

- K poset ideal in $2^{[n]}$ w.r.t. partial order inclusion.
- $K_j = \{ F \in K \mid |F| = j \}$.
- K is *shifted* if each K_j poset ideal in $\binom{[n]}{j}$ w.r.t.

$$\{a_1 < a_2 < \cdots < a_j\} \leq \{b_1 < b_2 < \cdots < b_j\} \iff \forall i : a_i \leq b_i$$

- $\deg_j(K, i) = \#\{ F \in K_j \mid i \in F \}$.
- $d_j(K) = (\deg_j(K,1), \deg_j(K,2), \ldots, \deg_j(K,n))$.
- Non-increasing if K shifted!

Shifted complexes have integral Laplacian spectra

Theorem (Duval and Reiner)

If K is a shifted simplicial complex then, for all j,

$$s'_j \approx d_j^T$$

In particular, all eigenvalues of L'_j are non-negative integers.

Conjecture (Duval and Reiner)

For any simplicial complex K, s_j is majorized by d^T.

Duval also proved that shifted s.c. and independence complexes of matroids satisfy a certain *spectral recursion*: $s(K)$ can be expressed in terms of $s(K-e)$ (deletion), $s(K/e)$ (contraction or link), and $s(K-e, K/e)$ (simplicial pair). Open question: which other s.c. fulfill this?

Questions

- Shifting?
- Extremality?
- Reading off ring-theoretic properties of the indicator algebra $\mathbb{R}[K]$ (or the Stanley-Reisner ring of K) from the Laplacian spectrum of K?

Multicomplexes

- $X = \{x_1, \ldots, x_n\}$.
- $[X]$ free abelian monoid, ordered by divisibility.
- $\mathbb{R}[X]$ polynomial ring.
- $M \subset [X]$ multicomplex iff finite poset ideal.
- $\mathbb{R}M$ vector space. Natural multiplication, iso with $\mathbb{R}[X]/I$, I artinian monomial ideal.

Boundary operator

- Björner and Vrećica:

$$\partial_d : \mathbb{R}[X]_d \to \mathbb{R}[X]_{d-1}$$
$$x_1^{a_1} \cdots x_n^{a_n} \mapsto \sum_{j=1}^{n} (-1)^{a_1+a_2+\cdots+a_{j-1}} r_j \frac{x_1^{a_1} \cdots x_n^{a_n}}{x_j}$$

with

$$r_j = \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases}$$

- Restricts to $\partial_d : \mathbb{R}M_d \to \mathbb{R}M_{d-1}$.
- On square-free monomials, ∂ is ordinary boundary operator.

Boundary operator

- Any monomial m can be uniquely written $m = p^2 q$ with q square-free.
- $\partial(p^2 q) = p^2 \partial(q)$.
- Put $M^{p^2} = \left\{ q \in M \middle| p^2 q \in M, q \text{ square-free} \right\}$, a simplicial complex.
- $\mathbb{R}M = \oplus_{p^2 \in M} p^2 \mathbb{R} M^{p^2}$.
- $\mathbb{R}M_\ell = \oplus_{p^2 \in M, 2|p| \leq \ell} p^2 \mathbb{R} M^{p^2}_{\ell - 2|p|}$.
- So,

$$H_\ell(M) \simeq \bigoplus_{p^2 \in M,\, 2|p| \leq \ell} H_{\ell - 2|p|}(M^{(p^2)}) \tag{1}$$

Laplacians on multicomplexes

- Dual boundary operator defined by

$$\begin{aligned} \partial^*_{d+1}(m) &= \sum_{j=1}^{n} (-1)^{a_1+\cdots+a_{j-1}} s_j x_j m, \\ s_j &= \begin{cases} 1 & \text{if } a_j \text{ is even and } x_j m \in M \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

- Define Laplacians as for simplicial complexes:

$$\begin{aligned} L'_d &= \partial_{d+1}\partial^*_{d+1} \\ L''_d &= \partial^*_d \partial_d \\ L_d &= L'_d + L''_d \end{aligned} \tag{3}$$

Laplacians on multicomplexes

- We have that

$$
\begin{aligned}
L'(p^2 q) &= p^2 L'(q) \\
L''(p^2 q) &= p^2 L''(q) \\
L(p^2 q) &= p^2 L(q)
\end{aligned}
\tag{4}
$$

- Define the spectra $\mathbf{s}'_i$, $\mathbf{s}''_i$, $\mathbf{s}^{tot}_i$, of the selfadjoint, nonnegative definite operators L'_i, L''_i, L_i to be the multiset of their (real and and nonnegative) eigenvalues. We will identify such a multiset with its weakly decreasing rearrangement, which is a partition, and we will, unless otherwise stated, identify such partitions that differ only in the number of zero parts.

Since everything splits...

Lemma

$$
\begin{aligned}
\mathbf{s}_i^{'}(M,\partial) &= \sum_{p^2 \in M,\, 2\deg(p) \le i} \mathbf{s}_{i-2\deg(p)}^{'}(M^{p^2},\partial) \\
\mathbf{s}_i^{''}(M,\partial) &= \sum_{p^2 \in M,\, 2\deg(p) \le i} \mathbf{s}_{i-2\deg(p)}^{''}(M^{p^2},\partial) \\
\mathbf{s}_i^{tot}(M,\partial) &= \sum_{p^2 \in M,\, 2\deg(p) \le i} \mathbf{s}_{i-2\deg(p)}^{tot}(M^{p^2},\partial) \qquad (5) \\
\mathbf{s}_i^{''}(M,\partial) &= \mathbf{s}_{i-1}^{'}(M,\partial) \\
\mathbf{s}_i^{tot}(M,\partial) &= \mathbf{s}_i^{'}(M,\partial) \cup \mathbf{s}_i^{''}(M,\partial) \\
\mathbf{s}_i^{'}(M,\partial) &= \mathbf{s}_i^{tot}(M,\partial) - \mathbf{s}_i^{''}(M,\partial)
\end{aligned}
$$

Shifted multicomplexes

Definition

A subcomplex $N \subseteq M$ is *shifted* (relative its support) if

$$x_j m \in N, \quad i < j, \quad x_i \in N \qquad \Longrightarrow \qquad x_i m \in N \tag{6}$$

Correspond to *strongly stable* artinian monomial ideals.

Lemma

If M is shifted, then so are all M_{p^2}, with the induced total ordering on the vertices in their supports.

Degree sequence

Definition

Let $N \subseteq M$ be a multicomplex. The *degree-sequence* $\mathbf{d}_k$ is the sequence

$$\mathbf{d}_k(N) = (d_1, d_2, d_3, \ldots, d_n) \tag{7}$$

where d_j denotes the number of monomials in N_k that are divisible by x_j.

Lemma

If N is shifted then $\mathbf{d}_k(N)$ is weakly decreasing, i.e. a partition.

Duval and Reiner Reformulated

Theorem

Suppose that M is shifted. Then

$$\mathbf{s}_k^{'} = \sum_{p^2 \in M,\, 2\deg(p) \leq k} \mathbf{d}_k^T(M_{p^2}) \tag{8}$$

In particular, the eigenvalues of L_d' are non-negative integers. Equivalently: let $\underline{b}$ be 1 if b is odd, and zero otherwise, and let

$$\underline{(\alpha_1, \ldots, \alpha_n)} = (\underline{\alpha_1}, \ldots, \underline{\alpha_n}) \tag{9}$$

Then

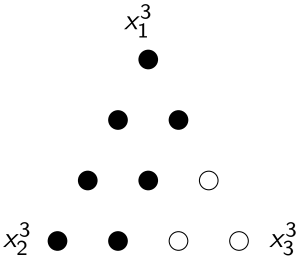
$$\mathbf{s}_k^{'\,T} = \sum_{\mathbf{x}^{\boldsymbol{\alpha}} \in M_k} \underline{\boldsymbol{\alpha}} \tag{10}$$

Example

Let

$$M_3 = \{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_3, x_1x_2x_3, x_1^2x_3\} \subset [x_1, x_2, x_3]_3,$$

as below:



Knowing M_3 lets us determine s'_3: the matrix of d_3, with respect to the basis of monomials of degree three and two ordered lexicographically, is

	x_1^3	$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	x_2^3	$x_2^2x_3$
x_1^2	1	-1	-1	0	0	0	0
x_1x_2	0	0	0	0	1	0	0
x_1x_3	0	0	0	0	-1	0	0
x_2^2	0	0	0	1	0	1	-1
x_2x_3	0	0	0	0	1	0	0
x_3^2	0	0	0	0	0	0	0

and PP^* has eigenvalues $3, 3, 3, 0, 0, 0$. We have that

$$\begin{aligned}(3,3,3)^T &= (3,3,3)\\ &= (1,0,0)+(0,1,0)+(0,0,1)+(1,0,0)+(1,1,1)+\\ &\qquad +(0,1,0)+(0,0,1).\end{aligned}$$

Questions

- Does Laplacian spectra work well with shifting?
- Spectral recursion?
- Are ring-theoretical properties of $\mathbb{R}M$ detectable from the Laplacian spectrum?
- Inequalities (majorization)?

$[N]$ as a multicomplex

- The multiplicative monoid $\mathbb{N}_+$ is free abelian on the set of primes.
- Any finite poset ideal in $\mathbb{N}_+$ (divisibility) is a multicomplex on a finite set of primes.
- In particular, $[N]$ is divisor-closed, so a multicomplex.
- For instance, $[6] = \{2^0 3^0 5^0, 2^1 3^0 5^0, 2^2 3^0 5^0, 2^0 3^0 5^1, 2^1 3^1 5^0\}$ is the multicomplex $\{(0,0,0),(1,0,0),(2,0,0),(0,0,1),(1,1,0)\}$.

Γ_n

- Call the associated multicomplex ring Γ_N. Then Γ_n can be realized as functions $f : [N] \to \mathbb{R}$, with multiplication modified Cauchy convolution

$$f * g(m) = \begin{cases} \sum_{k|m} f(k)g(m/k) & m \leq n \\ 0 & m > N \end{cases}$$

- It is also a quotient $C[x]/I_N$, I_N an artinian monomial ideal.
- Letting $N \to \infty$, we get $\Gamma = \varprojlim \Gamma_N$, the set of all arithmetical functions $f : \mathbb{N}_+ \to \mathbb{R}$ with Cauchy convolution

$$f * g(m) = \sum_{k|m} f(k)g(m/k)$$

This is the UFD $R[[x_1, x_2, x_3, \ldots]]$ (Cashwell-Everett).

Laplacian eigenvalues

- So, Γ_N is a natural truncation of a ring with number theoretic significance.
- Bi-graded Hilbert series easy to interpret.
- Betti numbers trickier (Eliahou-Kervaire), yield some interesting *problems* in analytic number theory (no new *results*).
- What about Laplacian eigenvalues?

Exact formulae

- Lucky us! $[N]$ is a shifted multicomplex!
- If $m = p_1^{a_1} \cdots p_r^{a_r}$ then $\log(m) = (a_1, a_2, \cdots)$.
- $\mathrm{sfp}(m)$ is square-free part.
- Then

$$\mathbf{s}_k^{'\,T} = \sum_{\substack{1 \le \ell \le N \\ \Omega(\ell) = k}} \log(\mathrm{sfp}(\ell))$$

- Put

$$\mathbf{s}_k^{'}(N)^T = (t_k^1(N), t_k^2(N), \ldots)$$

$$\mathbf{s}_k^{'}(N) = (s_k^1(N), s_k^2(N), \ldots)$$

Exact formulae

- Then

$$t_k^i(N) = \sum_{\substack{1<n\leq N \\ \Omega(n)=k \\ p_i|\mathrm{sfp}(n)}} 1$$

$$s_k^j(N) = \sum_{\{\, i \,|\, t_k^i(N)\geq j \,\}} 1$$

$$= \#\{\, \ell : \#\{\, 1 < n \leq N : \Omega(n) = k,\ p_\ell \,|\mathrm{sfp}(n) \,\} \geq j \,\}$$

- $s_1^1(N)$ is the number of primes $\leq N$.

What about $s_2^i(N)$?

- Let $Y_2(N)$ be square matrix, indexed by primes $\leq N$, a, b) entry is 1 if $p_a p_b \leq N$, $a \neq b$, 0 otherwise.
- Push the ones to the left edge to get $U_2(N)$, partion-shaped.
- This partition is $t_2(N)$, conjugate is $s_2(N)$.

Example: [50]

$$\begin{bmatrix} 0&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&0&0&0\\ 1&1&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Table: $Y_2(50)$ and $U_2(50)$. $\mathbf{t}_2 = (8,5,3,3,2,2,1,1,1)$, $\mathbf{s}_2 = (9,6,4,2,2,1,1,1)$.

Easier than composing boundary maps, then calculate characteristic polynomials, then finding roots...

Asymptotics

Pedestrian methods yield that

$$s_2^i(N) \sim \frac{N/p_i}{\mathcal{W}(N/p_i)}$$

where $\mathcal{W}$ is the Lambert W-function (solution to functional equation $z = \mathcal{W}(z)e^{\mathcal{W}(z)}$.