Term-orders and posets of compositions

Jan Snellman[1,2]

[1]Department of Mathematics
Stockholm University

[2]Department of Mathematics
University of Linköping

Mittag-Leffler, Apr 7, 2005

Slides at www.math.su.se/~jans/

Outline

$X = \{x_1, x_2, x_3, \ldots\}$, $X_n = \{1, x_2, \ldots, x_n\}$.
$X^* =$ free monoid on X, $X_n^* =$ free monoid on X_n.
$[X] =$ free abelian monoid on X, $[X_n] =$ free abelian monoid on X_n.
$\Delta(X) \subset [X]$ consists of square-free monomials, modified product
$u \times v = uv$ if square-free, 0 otherwise. $\Delta(X_n) \subset \Delta(X)$ restriction.
M will denote any of these monoids.

D divisibility order on $M = X^*$ or on $M = X_n^*$.

$$u \leq_D v \iff \exists w, t : v = tuw$$

D fulfills:

(i) $\forall v \in M \setminus \{1\} : 1 \leq v$,

(ii) $\forall u, v, w, t \in M : \; u \leq v \implies tuw \leq tvw$,

So (X^*, D) and (X^*, D) are partially ordered monoids, pomonoids.

D divisibility order on $M = [X]$ or on $M = [X_n]$.

$$u \leq_D v \iff \exists w : v = uw$$

D fulfills:

(i) $\forall v \in M \setminus \{1\} : 1 \leq v$,

(ii) $\forall u, v, w \in M : \; u \leq v \implies uw \leq vw$,

So $([X], D)$ and $([X_n], D)$ are partially ordered monoids.

D divisibility order on $M = \Delta(X)$ or on $M = \Delta(X_n)$.

$$u \leq_D v \iff \exists w : v = uw$$

D fulfills:

(i) $\forall v \in M \setminus \{1\} : 1 \leq v$,

(ii) $M = \Delta(X)$: $\forall u, v, w \in M : (u \leq v) \wedge uv \neq 0) \wedge (uw \neq 0) \implies uw \leq vw$.

So $(\Delta(X), D)$ and $(\Delta(X_n), D)$ are partially ordered monoids.

Any multiplicative total extension of D, i.e. a total order $\succeq$ on M satisfying these conditions, is called a term order. By Higman's lemma, they are *well-orders* for finitely many variables, i.e. there are no infinite descending chains

$$u_1 \succ u_2 \succ u_3 \succ u_4 \succ \cdots$$

The term order $\succeq$ is standard if

$$x_1 \prec x_2 \prec x_3 \prec x_4 \prec \cdots$$

Standard term orders are well-orders, even for infinitely many variables.

Any multiplicative partial order on $[X_n] \simeq \mathbb{N}^n$ extends uniquely to the difference group $\mathbb{Z}^n$ by

$$\mathbf{x}^{\boldsymbol{\alpha}} \leq \mathbf{x}^{\boldsymbol{\beta}} \iff \boldsymbol{\alpha} \leq \boldsymbol{\beta} \iff \mathbf{0} \leq \boldsymbol{\beta} - \boldsymbol{\alpha}.$$

Furthermore, it extends uniquely to $\mathbb{Q}^n$, and then to $\mathbb{R}^n$.
Conversely, any multiplicative partial order on $\mathbb{R}^n$ restricts to a multiplicative partial order on $\mathbb{Z}^n$.
A non-zero weight vector $\mathbf{v} \in \mathbb{R}^n$ yields a multiplicative partial order on $\mathbb{R}^n$ by

$$\boldsymbol{\alpha} \geq \boldsymbol{\beta} \iff \langle \boldsymbol{\alpha}, \mathbf{v} \rangle \geq \langle \boldsymbol{\beta}, \mathbf{v} \rangle \iff \langle \boldsymbol{\alpha} - \boldsymbol{\beta}, \mathbf{v} \rangle \geq 0.$$

If $\mathbf{v_1}, \ldots, \mathbf{v_r} \in \mathbb{R}^n$ are non-zero weight vectors, they define a multiplicative partial order by $\boldsymbol{\alpha} \geq \mathbf{0}$ iff

- either all $\langle \boldsymbol{\alpha}, \mathbf{v_i} \rangle$ are zero, or
- the first non-zero such number is positive.

The multiplicative partial order induced by $\mathbf{e_1}, \ldots, \mathbf{e_n}$ is a term order, called the *lexicographic* term order. We have that

$$x_1 \leq_{\text{lex}} x_2 \leq_{\text{lex}} \cdots \leq_{\text{lex}} x_n,$$

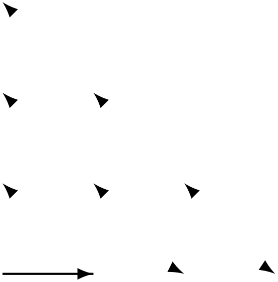
so it is not standard; however, by permuting the variables we get a a standard lex order.
Refining the multiplicative partial order induced by $\mathbf{e_1} + \cdots + \mathbf{e_n}$ by $\mathbf{e_1}, \ldots, \mathbf{e_n}$ we get the *total degree, then lexicographic* term order.

Theorem (Robbiano et al)

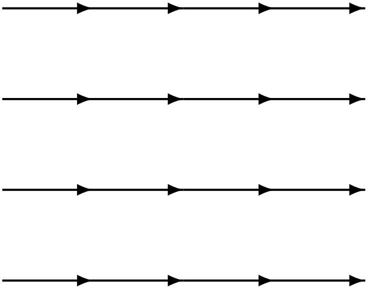
Any term order $\succ$ on $[X_n]$ is given by a tuple of at most n weight vectors. In other words, there is a real n times n matrix A such that

$$\mathbf{x}^{\boldsymbol{\alpha}} \succ \mathbf{x}^{\boldsymbol{\beta}} \iff A\boldsymbol{\alpha} \geq_{\mathrm{lex}} A\boldsymbol{\beta}.$$

The possible order types of $\succ$ are $\omega, \omega^2, \ldots, \omega^n$. The term orders with order type ω^n are precisely the $n!$ lexicographic orders.



Total degree, then lex, with $x_1 < x_2$, has order type ω.



Lex, with $x_1 < x_2$, has order type ω^2.

Intersection of standard term orders

$M = [X]$ **or** $M = [X_n]$

$$D = \bigcap_{\ell \text{ term order on } M} \ell$$

$$\mathcal{Y} = \bigcap_{\ell \text{ standard term order on } M} \ell$$

D on $[X_2]$:

$$x_1^2 \qquad x_1 x_2 \qquad x_2^2$$

$$x_1 \qquad x_2$$

$$1$$

Impose $x_1 < x_2$, and all consequences, e.g. $x_1^2 < x_1 x_2$. Get:

x_1x_2 x_1^3

x_2 x_1^2

x_1

1

Covering rels of $\mathcal{Y}$

Partially defined operators on $[X]$ and on $[X_n]$

$m = x_1^{a_1} \cdots x_s^{a_s}$.
$L(m) = x_1 m$, defined everywhere.
$U_j(m) = \frac{x_{j+1}}{x_j} m$, defined when $a_j > 0$ (and $j < n$).

The covering relations of $\mathcal{Y}$ are precisely $m \lessdot L(m)$ and $m \lessdot U_j(m)$.

Strongly stable (Borel) ideals

Filters w.r.t. $D \iff$ monomial ideals

Filters w.r.t. $\mathcal{Y} \iff$ strongly stable monomial ideals

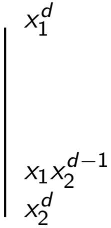
Strongly stable ideals are fixed under the action of the Borel subgroup of upper-triangular matrices. Important example: generic initial ideals.

Borelfixed subsets

Restrict $\mathcal{Y}$ to $[X_n]_d$, monomials of total degree d.

$$\text{Filters w.r.t. } \mathcal{Y} \quad \Longleftrightarrow \quad \text{Borel-fixed subsets.}$$

Important to know the number of such subsets of given cardinality (study of minimal free resolutions, algebraic geometry).



The Hasse diagram for the strongly stable partial order for $n = 2$.

The Hasse diagram for the strongly stable partial order for $n = 3$.

x_1^d

x_2^d

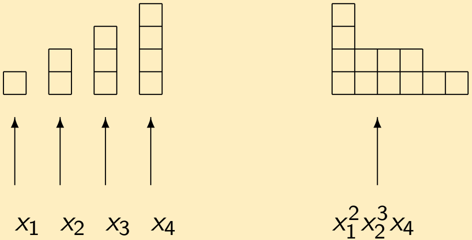
x_3^d

Bijection with Young's lattice

$\mathbf{f_j} = \mathbf{e_1} + \cdots + \mathbf{e_j}$. Order-isomorphism

$$([X], \mathcal{Y}) \xrightarrow{log} \mathbb{N}^{\omega} \to Y$$
$$x_1^{a_1} \cdots x_s^{a_s} \mapsto (a_1, \ldots, a_s) \mapsto a_1\mathbf{e_1} + \cdots + a_s\mathbf{e_s}$$

Pictorial representation



$\mathcal{Y}$ is Young

$$
\begin{aligned}
([X], \mathcal{Y}) &\simeq Y \text{ Young's lattice} \\
([X_n], \mathcal{Y}) &\simeq \text{ At most } n \text{ rows} \\
([X_n]_d, \mathcal{Y}) &\simeq \text{ At most } n \text{ rows, exactly } d \text{ columns} \\
&\simeq \text{ At most } n \text{ rows, at most } d-1 \text{ columns}
\end{aligned}
$$

$[X_n]_d, \mathcal{Y})$ has following properties:

- Sperner
- Rank-symmetric
- Rank-unimodular
- Rank numbers given by coeffs of q-binomial polynomials

1
1
2
2
3
3
3
2
2
1
1

$$\frac{(1-q^6)(1-q^7)}{(1-q)(1-q^2)} = q^{10} + q^9 + 2q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

Filters in $([X_3]_d, \mathcal{Y})$

Following iso:

1. Filters in $([X_3]_d, \mathcal{Y})$
2. Partitions into distinct parts $\leq d+1$,
3. $(\Delta([X_3]), \mathcal{Y})$, $\mathcal{Y}$ restricted to square-free monomials.

Definition

$\leq$ is a standard term order on X^* (or on X_n^*) iff

1. $\leq$ is a total order on X^*
2. $1 < x_1 < x_2 < x_3 < \cdots$,
3. $u \leq v \implies sut \leq svt$.

Higman's lemma $\implies$ $\leq$ well-order. Skip classification

Non-commutative term orders

Classification

X_1^*: only $1 < x_1 < x_1^2 < x_1^3 < \cdots$, order type ω.
X_2^*: Possible order types are

- ω, e.g. total degree, then lex.
- ω^2, e.g. lex.
- ω^ω, only 2 such orderings!

X_n^*: maximal order type $\omega^{\omega^{n-1}}$.
X^*: maximal order type ω^{ω^ω}.

Skip Kachinuki ordering

$u, v \in X_2^*$, u has a occuring x_2, v has b occuring x_2. Write

$$u = yu', \quad v = yv', \quad u, v \text{ has no common non-empty prefix.}$$

Kachinuki ordering

$u > v$ iff either

1. $a > b$, or
2. $a = b$ and $u \neq v$ and $v' = 1$ (empty string) or $v' = x_2 z$, $z \in X_2^*$.

The reversal of this is the only other standard term order of order type ω^ω.

Intersection of standard term orders

$M = X^*$ or $M = X_n^*$

$$D = \bigcap_{\ell \text{ term order on } M} \ell$$

$$\mathfrak{N} = \bigcap_{\ell \text{ standard term order on } M} \ell$$

D on X_2^*:

$x_1 x_2$ $x_2 x_1$ x_2^2

x_1^2

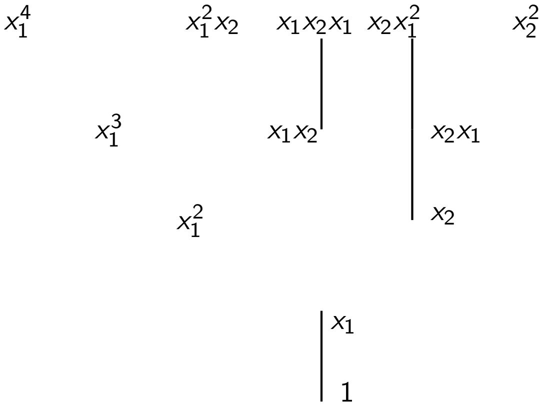
x_1 x_2

1

Impose $x_1 < x_2$, and all consequences, e.g. $x_1^2 < x_1 x_2$. Get:

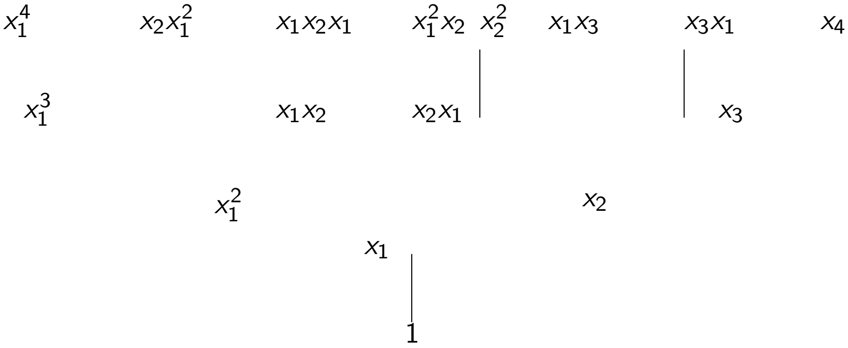
Intersection of standard term orders

$\mathfrak{N}$ on X_2^*



Intersection of standard term orders

$\mathfrak{N}$ on X^*



▸ Label edges in Hasse diagram

Partially defined operators on X^* and on X_n^*

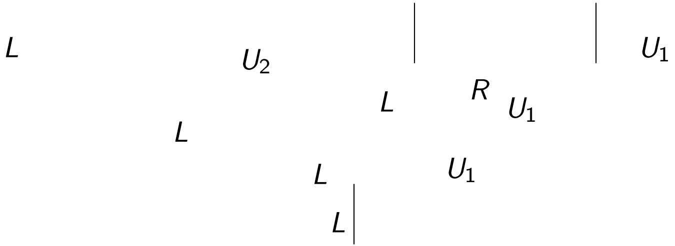
$m = x_{a_1} \cdots x_{a_s}$.
$L(m) = x_1 m$, defined everywhere. $R(m) = m x_1$, defined everywhere.
$U_j(m) = x_{a_1} \cdots x_{a_{j-1}} x_{a_j + 1} x_{a_{j+1}} \cdots x_{a_s}$, defined when $j \leq s$ (and $a_j < n$).

The covering relations of $\mathfrak{N}$ are precisely $m \lessdot L(m)$ and $m \lessdot R(m)$ and $m \lessdot U_j(m)$.

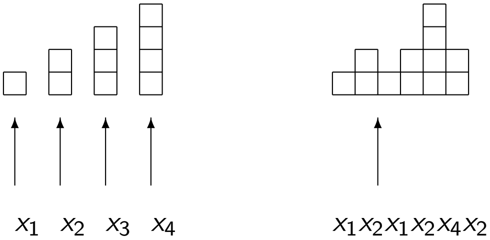
$\langle L, R, U_1, U_2, \ldots \rangle$ free (non-comm) semigroup.
Partial left action on $\mathfrak{N}$ by

$$\begin{aligned} WR.m &= W.R(m) \\ WL.m &= W.L(m) \\ WU_j.m &= W.U_j(m) \end{aligned} \quad (1)$$

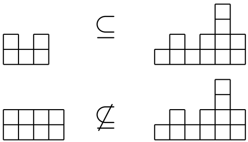
If $m = x_1^k$ then $L.m = R.m$, otherwise action by different letters give different result. Convention: R not allowed to act on x_1^k.
Can label edges in Hasse diag according to type of covering rel, get:



▸ Label vertices in Hasse diagram



Compositions ordered by inclusion of diagrams

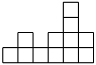
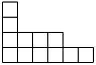


Definition

An r-multiranking on a poset P is a map $\phi : P \to \mathbb{N}^r$ such that

$$u \lessdot v \implies \phi(u) \lessdot \phi(v)$$

The Young lattice is ω-multiranked, and so is $\mathfrak{N}$.

Multirank of  is  $= (4, 2, 2, 2, 1, 1) \in \mathrm{Young}$. Rank is the number of boxes, i.e. 12.

$$\text{Multirank gf} = (1 - \sum_{i=1}^{\infty} \prod_{j=1}^{i} t_j)^{-1}$$

$$\text{gf} = \frac{1-t}{1-2t}$$

Enumeration of saturated chains in $\mathfrak{N}$

d'après Bergeron, Bousquet-Mélou, et Dulucq

$\alpha \leq \beta$ in $\mathfrak{N}$.
$\gamma = (p_0, p_1, \ldots, p_s)$ saturated chain from α to β, of length s:

$$\alpha = p_0 \lessdot p_1 \lessdot p_1 \lessdot \cdots \lessdot p_s = \beta$$

Example: $\gamma = (121, 1121, 1122, 2122, 2222, 2322)$.
Correspond to tableau on β, coding in which order boxes are added:

	5		
3	4	0	2
1	0	0	0

Defines width and height of γ.

$\gamma = (\alpha = p_0, \ldots, p_n = \beta)$, with $\beta = (a_1, \ldots, a_k)$.
$v(\gamma) = v(\beta) = t_1^{a_1} \cdots t_k^{a_k}$.

$$f_k^{\alpha} = \sum_{\gamma \text{ width } k, \text{ from } \alpha} v(\gamma).$$

Explicit description of cover rel in terms of L, R, U_j gives recurrence relation for f_k^{α}.

Definition

We say that the compositions 11...1 are all-one, “a.o”.
We define $\Lambda(h)$ to be the result of performing the substitutions $t_i \mapsto t_{i+1}$ on h.

Theorem

Let $\alpha \in \mathfrak{N}$ have width r. Then

$$f_k^\alpha = \begin{cases} 0 & \text{if } k < r \\ A + v(\alpha) & \text{if } k = r \\ A + B + C & \text{if } k > r,\ \alpha \text{ not a.o} \\ A + B + C - t_1 t_2 \cdots t_k & \text{if } k > r,\ \alpha \text{ a.o} \end{cases} \tag{2}$$

$$A = (t_1 + t_2 + \cdots + t_k) f_k^\alpha$$

$$B = t_1 \Lambda(f_{k-1}^\alpha)$$

$$C = t_k f_{k-1}^\alpha$$

$$f_0^{()} = 1$$

$$f_1^{()} = \frac{t_1}{1 - t_1}$$

$$f_2^{()} = \frac{t_1 t_2 (1 - t_1 t_2)}{(1 - t_1)(1 - t_2)(1 - t_1 - t_2)}$$

$$f_3^{()} = t_1 t_2 t_3 \times (1 - t_1)^{-1} (1 - t_2)^{-1} (1 - t_3)^{-1} \times$$
$$(1 - t_1 - t_2)^{-1} (1 - t_2 - t_3)^{-1} (1 - t_1 - t_2 - t_3)^{-1} \times \text{ junk}$$

Theorem

For each k,

$$f_k(t_1,\ldots,t_k) = \frac{t_1\cdots t_k}{\prod_{i=1}^{k}\prod_{j=i}^{k}(1-t_i-t_{i+1}-\ldots-t_j)}\tilde{f}_k(t_1,\ldots,t_k) \tag{3}$$

where $\tilde{f}_k$ is a polynomial.

Definition

$a_{n,k}$: number of standard paths of width k and length n.

$$L_k(t) = \sum_{n\geq 0} a_{n,k} t^n$$

Note: $L_k(t) = f_k(t, \ldots, t)$.

$$\begin{aligned} L_1(t) &= \frac{t}{1-t} \\ L_2(t) &= \frac{t^2(1+t)}{(1-t)(1-2t)} \\ L_3(t) &= \frac{t^3(1+5t-2t^2)}{(1-t)(1-2t)(1-3t)} \end{aligned} \tag{4}$$

Theorem

$$L_k(t) = \frac{t^k \tilde{L}_k(t)}{\prod_{i=1}^{k}(1-it)} \tag{5}$$

where $\tilde{L}_k(t)$ *is a polynomial of degree* $k-1$ *with* $\tilde{L}_k(1) = 2^{k-1}$.

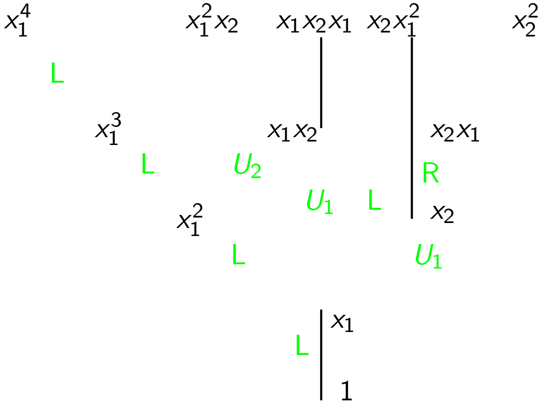
Corollary

For a fixed k,

$$a_{n+k,k} \sim \frac{k^{k-1}}{(k-1)!} k^n \qquad \text{as } n \to \infty \tag{6}$$

Labeled enumeration of chains of fixed width

Labeling of edges



If $m \in \mathfrak{N}$ and $W = W_r W_{r-1} \dots W_1$ is a word in $\langle L, R, U_j \rangle$ which is admissible for m, we give the corresponding chain $\gamma = (m, W_1.m, W_2 W_1.m, \dots, W.m)$ non-commutative weight

$$V(\gamma) = v(\gamma) W \tag{7}$$

Example:

$$V((), 1, 11, 12) = t_1 t_2 U_2 LL$$

▸ Label edges in Hasse diagram

Definition

$$F_k^\alpha = \sum_\gamma V(\gamma) \tag{8}$$

$$F_K^\alpha = \begin{cases} 0 & \text{if } k < s \\ A + v(\alpha) & k = s \\ A + B + C & \text{if } k > s, \alpha \text{ not a.o} \\ A + B + C - D & k > s, \alpha \text{ a.o} \end{cases}$$

$$A = (t_1 U_1 + \cdots + t_k U_k) F_k^\alpha$$

$$B = t_1 L \cdot \Lambda(F_{k-1}^\alpha)$$

$$C = t_k R \cdot F_{k-1}^\alpha$$

$$D = R \cdot L^{k-1} v(\alpha)$$

Labeled enumeration of chains of fixed width

$\alpha = (2)$

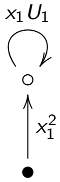
$$
\begin{aligned}
F_0^{(2)} &= 0 \\
F_1^{(2)} &= (1 - x_1 U_1)^{-1} x_1^2 \\
F_2^{(2)} &= (1 - x_1 U_1 - x_2 U_2)^{-1} \times \\
&\quad \left[x_1 L (1 - x_2 U_1)^{-1} x_2^2 + x_2 R (1 - x_1 U_1)^{-1} x_1^2 \right]
\end{aligned}
$$

Labeled enumeration of chains of fixed width

Non-commutative rational series in finitely many variables are *recognizable*: coefficients correspond to the labels of walks from a start node to an end node in a certain labeled digraph.
Example:

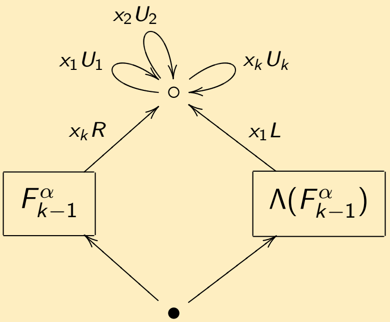
$$F_1^{(2)} = (1 - x_1 U_1)^{-1} x_1^2 = x_1^2 + x_1^3 U_1 + x_1^4 U_1^2 + \cdots$$

corresponds to paths from $\bullet$ to $\circ$ in the following digraph:

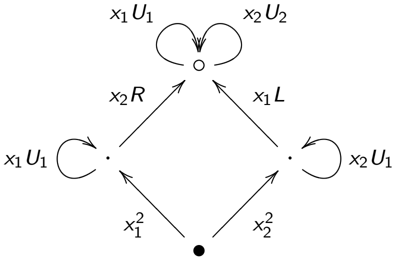


Theorem

Suppose that α is not all-ones. Then a digraph for F_k^α, which enumerates saturated chains of widht k in $\mathfrak{N}$, starting from α, by walks from $\bullet$ to $\circ$, is obtained from the one for F_{k-1}^α by



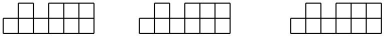
Digraph for $F_2^{(2)}$:



▸ Digraph for $F_3^{(2)}$ ▸ Skip enumeration of chains of height at most two

$c_{i,j}$ number of saturated chains from () to composition with i one's and j two's. Described by tableaux of height ≤ 2.
Such a tableaux can be obtained

1. from a tableau with $i-1$ parts of size 1 and j parts of size 2, by adding a part of size 1 to the left,
2. or from a tableau with $i-1$ parts of size 1 and j parts of size 2, by adding a part of size 1 to the right,
3. or from a tableau with $i+1$ parts of size 1 and $j-1$ parts of size 2, by adding a box to a part of size 1.



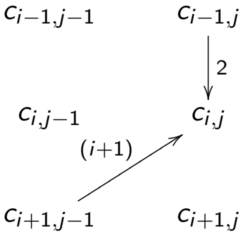
Enumeration of chains in $(X_2^*, \mathfrak{N})$

Recurrence

We get the recurrence

$$c_{i,j} = 2c_{i-1,j} + (i+1)c_{i+1,j-1} - \delta_j^0 \tag{9}$$

where δ_i^j is the Kronecker delta.



Boundary values $c_{n,0} = 1$ for $n \geq 0$.

For small values of i, j, $c_{i,j}$ is

j / i	0	1	2	3	4	5
0	1	1	4	30	336	5040
1	1	4	30	336	5040	95040
2	1	11	138	2184	42480	986040
3	1	26	504	10800	265320	7447440
4	1	57	1608	45090	1368840	45765720
5	1	120	4698	167640	6174168	242686080
6	1	247	12910	572748	25192440	1151011680
7	1	502	33924	1834872	95091360	4999942080
8	1	1013	86172	5588310	337239840	-

Theorem

Put

$$P_k(x) = \sum_{n=0}^{\infty} c_{n,k} x^n \tag{10}$$

Then $P_0(x) = (1-x)^{-1}$ *and*

$$P_k(x) = \frac{\frac{d}{dx} P_{k-1}(x)}{1-2x} \tag{11}$$

$$\begin{aligned}
P_0(x) &= (1-x)^{-1} \\
P_1(x) &= (1-x)^{-2}(1-2x)^{-1} \\
P_2(x) &= 2!(1-x)^{-3}(1-2x)^{-3}(2-3x) \\
P_3(x) &= 3!(1-x)^{-4}(1-2x)^{-5}(5-14x+10x^2x) \\
P_4(x) &= 4!(1-x)^{-5}(1-2x)^{-7}(14-56x+76x^2-35x^3)
\end{aligned} \tag{12}$$

and in general

$$P_k(x) = k!(1-x)^{-1-k}(1-2x)^{1-2k}Q_k(x) \tag{13}$$

where $Q_k(x)$ is a primitive polynomial of degree $k-1$, with $Q_k(1) = (-1)^{k+1}$.

Theorem

Put

$$P(x,y) = \sum_{i,j \geq 0} c_{i,j} x^i \frac{y^j}{j!} \tag{14}$$

Then

$$P(x,y) = \frac{2}{1 + \sqrt{1 - 4(y + x - x^2)}} \tag{15}$$

Proof.

We get from the recurrence relation (10) that

$$(1-2x)\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} \tag{16}$$

□

Theorem

With the notations above,

$$c_{0,n} = \frac{(2n)!}{(n+1)!} = n!\,C_n$$

$$c_{1,n} = c_{0,n+1} = \frac{(2(n+1))!}{(n+2)!}$$

$$c_{2,n} = \frac{1}{2}c_{0,n+2} - c_{0,n+1} = \frac{1}{16}\frac{\left(2\,n^2+6\,n+3\right)2^{2\,n+6}\Gamma(n+3/2)}{(n+3)\sqrt{\pi}\,(n+2)} \qquad (17)$$

Proof.

$$P(0, y) = \frac{2}{1 + \sqrt{1 - 4y}} \tag{18}$$

is the ordinary generating function for the Catalan numbers. □

Summary

- $\mathfrak{N}$ is a candidate for non-commutative Young poset
- $\mathfrak{N}$ catalogues standard non-commutative term orders
- Saturated chains in $\mathfrak{N}$ correspond to generalized tableaux. Some types (fixed widht, height ≤ 2) have been enumerated.

- To do
  - More refined enumeration, mimicking advanced techniques from Bergeron, Bousquet-Mélou, et Dulucq: coding as labeled binary trees.
  - Möbius function?
  - Probability that $x_1 x_2 x_1 > x_2^2$?

For Further Reading I

François Bergeron, Mireille Bousquet-Mélou, and Serge Dulucq.
Standard paths in the composition poset.
Ann. Sci. Math. Québec, 19(2):139–151, 1995.

Jan Snellman.
Standard paths in another composition poset.
Electron. J. Combin., 11(1):Research Paper 76, 8 pp. (electronic), 2004.

Jan Snellman.
A poset classifying non-commutative term orders.
In *Discrete models: Combinatorics, Computation, and Geometry*, Discrete Mathematics and Theorethical Computer Science Proceedings **AA (DM-CCG)**, pages 301–314, 2001.

For Further Reading II

Matthias Aschenbrenner and Wai Yan Pong
Orderings of monomial ideals
Fundamenta Mathematicae, 181:27–74, 2004.

Jan Snellman.
On some partial orders associated to generic initial ideals.
Séminaire Lotharingien de Combinatoire, B43h, 2000.

Digraph for $F_3^{(2)}$: