A poset classifying non-commutative term orders

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Strongly stable ideals

$S = \mathbb{C}[x_1, \ldots, x_n]$, $V = S_1$. Then $S \simeq S(V)$, the symmetric algebra on V.

- GL(V) = the general linear group $\simeq n \times n$ matrices
- $U \subset$ GL(V) upper triangular matrices
- Diag $\subset$ GL(V) diagonal matrices

These groups act on S by $g(x_i) = \sum_{j=1}^{n} g_{ij} x_j$. If $I \subset S$ is a homogeneous ideal, then Diag fixes I iff I is a monomial ideal. GL(V) fixes I iff $I = S_{\geq d}$ for some d. U fixes I if I is a special kind of monomial ideal, a strongly stable ideal:

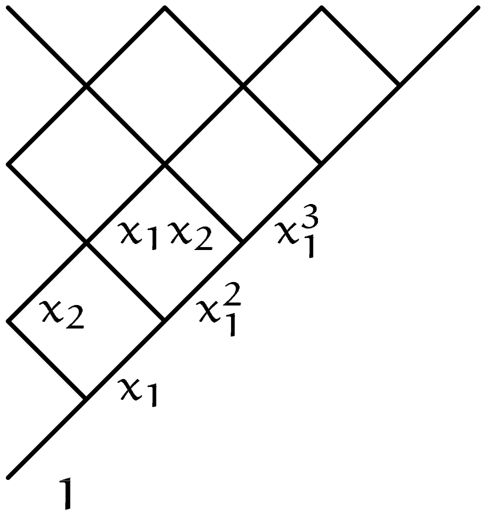
$$I \ni x_1^{a_1} \cdots x_n^{a_n},\ a_i > 0,\ i \leq j \leq n \quad \Longrightarrow \quad \frac{x_j}{x_i} x_1^{a_1} \cdots x_n^{a_n} \in I \tag{1}$$

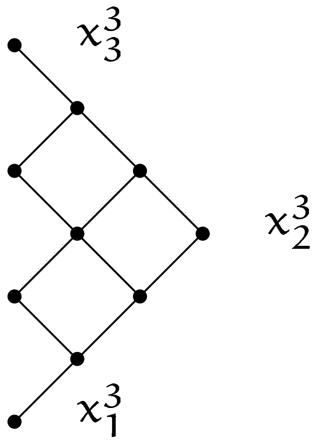
The associated poset

Let $[x_1, \ldots, x_n]$ denote the free abelian monoid. We regard it as the subset of monomials in $\mathbb{C}[x_1, \ldots, x_n]$. A monomial ideal I determines, and is determined by, the monoid ideal $I \cap [x_1, \ldots, x_n]$. If we give the monoid $[x_1, \ldots, x_n]$ the divisibility partial order D, then monoid ideals are precisely the filters w.r.t D.

Similarly, I is a strongly stable ideal iff $I \cap [x_1, \ldots, x_n]$ is a filter w.r.t the strongly stable partial order $\mathfrak{C}$ on $[x_1, \ldots, x_n]$.

Left: $\mathfrak{C}$ for $n = 2$, Right: $\mathfrak{C}$ restricted to monomials of degree 3, $n = 3$.



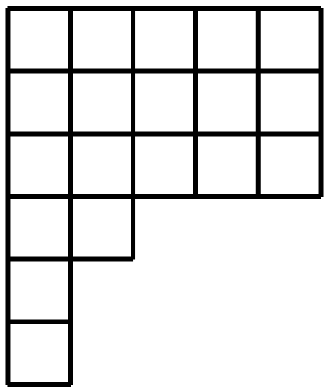


Relation to the Young lattice

It turns out that $\mathfrak{C}$ is isomorphic to the set of Ferrers diagrams with at most n columns, ordered by inclusion. The isomorphism is as follows:

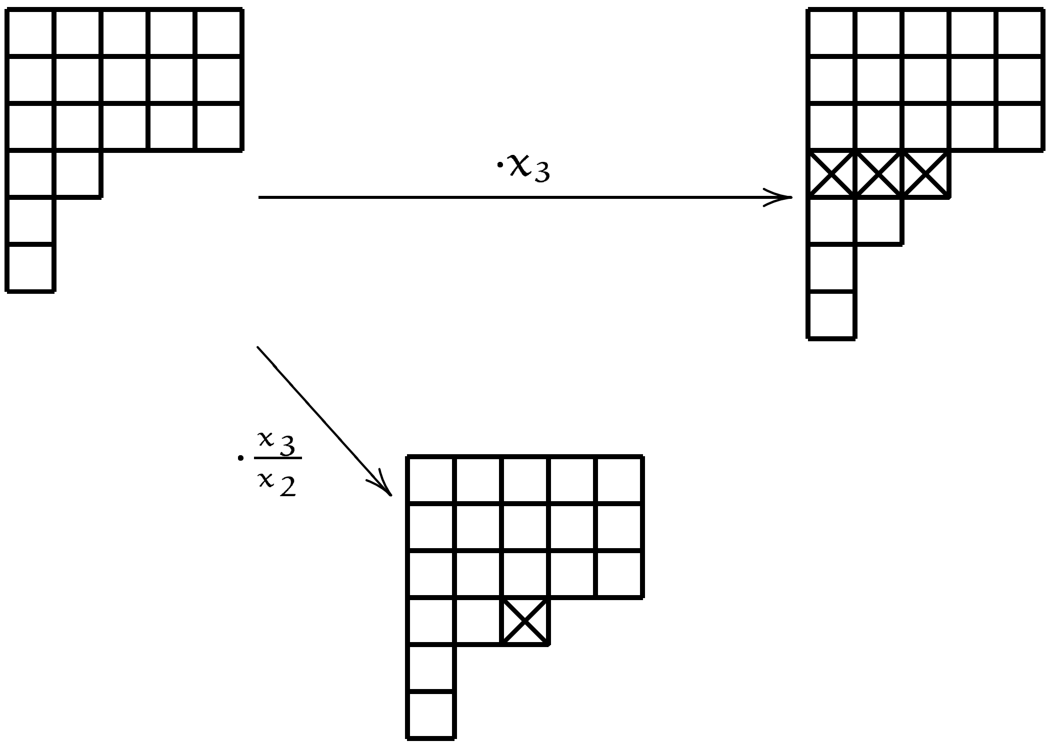
$$x_1^{a_1}\cdots x_n^{a_n} \mapsto (\underbrace{n,\ldots,n}_{a_n},\ldots,\underbrace{1,\ldots,1}_{a_1}) \tag{2}$$

The number of rows correspond to the total degree of the monomial. Below: the Ferrers diagram corresponding to $x_1^2 x_2 x_5^3$.



Relation to the Young lattice, cont

One can show that (2) is order-preserving with order-preserving inverse. We illustrate this below with $x_1^2 x_2 x_5^3 \cdot x_3$ and $x_1^2 x_2 x_5^3 \cdot \frac{x_3}{x_2}$.



Relation to term orders

A term order $\geq$ on $[x_1, \ldots, x_n]$ is a total well order that respects the multiplication, so that

$$1 \leq m \text{ for all } m \in [x_1, \ldots, x_n] \tag{3}$$

$$m \geq m' \implies tm \geq tm' \tag{4}$$

We say that $>$ is standard if $x_1 \leq x_2 \leq \cdots \leq x_n$.

The intersection of all standard term orders is precisely the strongly stable partial order $\mathfrak{C}$. In other words, if two monomials m, m' form an antichain w.r.t. $\mathfrak{C}$ (for instance, x_2^2 and $x_1 x_3$) then there are two standard term orders $\geq_1$, $\geq_2$ such that $m \geq_1 m'$ but $m \leq_2 m'$ (e.g. degrevlex and deglex).

The non-commutative analogue

$X^* =$ the free (non-commutative) monoid on $x_1, \ldots, x_n$,
$T = \mathbb{C}[X^*] =$ the free non-commutative polynomial ring,
$V = T_1$. Then $T \simeq T(V)$, so $\mathrm{GL}(V)$ acts on T. We call a monomial ideal $I \subset T$ strongly stable iff it is fixed by $\mathfrak{U}$, that is, if

$$I \ni m = x_{a_1} \cdots x_{a_r},\ a_i \leq j \leq n \implies \frac{x_j}{x_{a_i}} m \in I \tag{5}$$

Note: Diag and $\mathfrak{U}$ also fixes some non-monomial ideals!

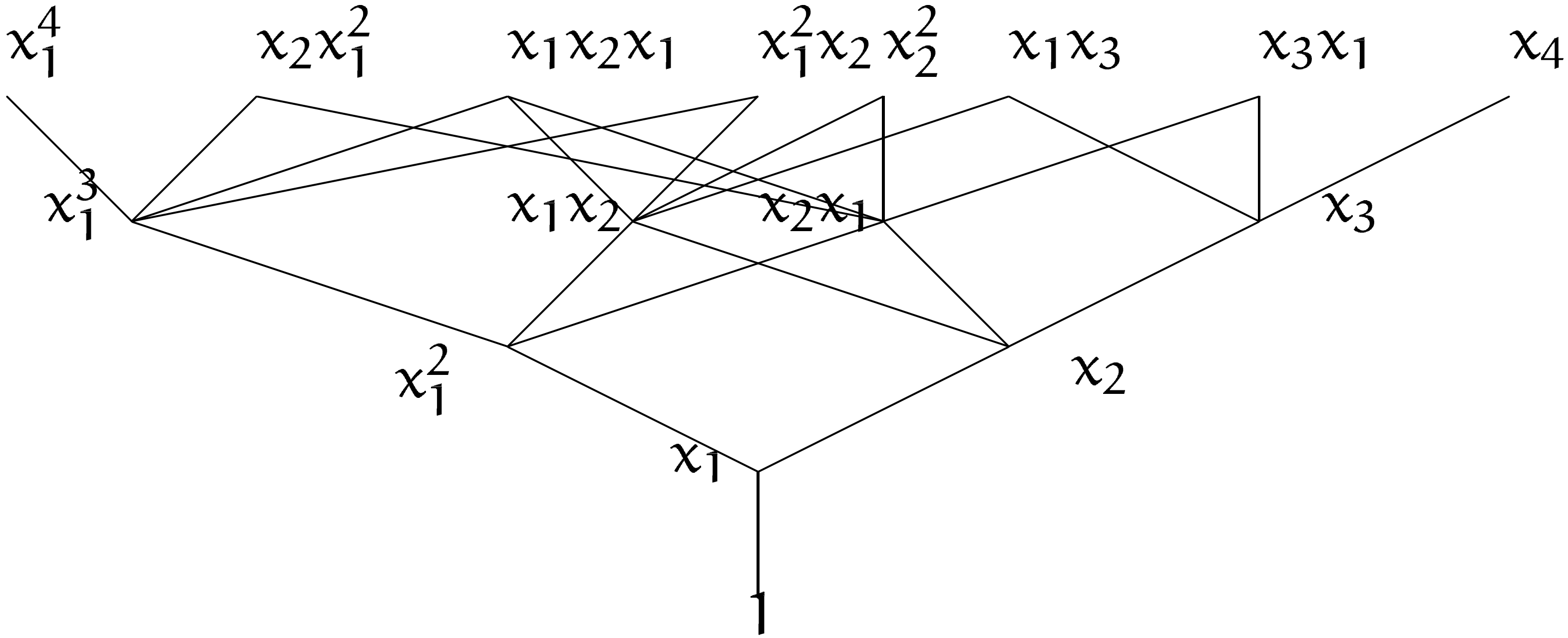
Parallel to the commutative case, strongly stable monomial ideals correspond to filters in the partially ordered set $(X^*, \mathfrak{N})$, where $\mathfrak{N}$ is the analogously defined strongly stable poset.

Description of the strongly stable poset

If $m, m' \in X^*$, then m is bigger than m' w.r.t $\mathfrak{N}$ if m can be obtained from m' by repeated applications of

1. $u \mapsto sut$, $s, t \in X^*$,
2. $tx_i s \mapsto tx_j s$, $i < j$,

The Hasse diagram of $\mathfrak{N}$



Clearly, $\mathfrak{N}$ is not a lattice.

Non-commutative term orders

A well-order $\geq$ on X^* is a term order if it respect the multiplication, i.e. if

$$1 \leq m \text{ for all } m \in X^* \tag{6}$$

$$m \geq m' \implies tms \geq tm's \tag{7}$$

We say that $>$ is standard if $x_1 \leq x_2 \leq \cdots \leq x_n$.
The intersection of all standard term orders is precisely the strongly stable partial order $\mathfrak{N}$.

Can this be used to understand the possible order types of sorted term orders? Already for 2 vars, there are strange (Kachinuki) term orders with order type ω^ω.

Multi-ranking

A poset $(P, \geq)$ is r-multi-ranked if there is an order-preserving map $\phi : P \to \mathbb{N}^r$ (where $\mathbb{N}^r$ is given the natural product order) such that $m \gtrdot m' \implies \phi(m) \gtrdot \phi(m')$. Thus a 1-ranked poset is nothing but a ranked poset in the ordinary sense, and all r-ranked posets are s-ranked, for $1 \leq s \leq r$.

The composition ϕ_c is an n-ranking of $([x_1, \ldots, x_n], \mathfrak{C})$.

$$[x_1, \ldots, x_n] \xrightarrow{\text{log}} \mathbb{N}^n \xrightarrow{F} \mathbb{N}^n \tag{8}$$

$\log(x_1^{a_1} \cdots x_n^{a_n}) = (a_1, \ldots, a_n)$, and F is the $\mathbb{Z}$-linear extension of $F(\mathbf{e}_i) = \sum_{j=1}^{i} \mathbf{e}_j$. $\phi(X^*) =$ decreasing vectors $=$ Ferrers diagrams with at most n rows. So ϕ_c is (when followed by a transposition of Ferrers diagrams) the previous bijection between $([x_1, \ldots, x_n], \mathfrak{C})$ and the set of Ferrers diagrams with at most n columns.

$(X^*, \mathfrak{M})$ ***is*** n***-ranked.***

An n-ranking ϕ_n is defined by the composition

$$X^* \xrightarrow{\sigma} [x_1, \ldots, x_n] \xrightarrow{\phi_c} \mathbb{N}^n \tag{9}$$

where $\sigma(x_{b_1} \cdots x_{b_r}) = x_1^{a_1} \cdots x_n^{a_n}$ is the corresponding commutative word, i.e. a_j is the number of ℓ such that $b_\ell = j$. Thus if $m \gtrdot m'$ then $\phi_n(m) \gtrdot \phi_n(m')$. Not conversely: $x_2 x_1^2$ and $x_1 x_2$ form an anti-chain, $(3, 1) \gtrdot (2, 1)$.

When does $m \gtrdot m'$?

The cover(t) relation EXPOSED!

$m = x_{i_1} \cdots x_{i_d}$, $N = \max(\{i_1, \ldots, i_d\})$, a_i the number of occurrences of x_i in m.
m is covered by the following words:

- $x_1 m$ and $m x_1$,
- The a_1 words obtained by replacing one occurrence of x_1 by x_2, the a_2 words obtained by replacing one occurrence of x_2 by x_3, and so on, up to and including the a_{n-1} words obtained by replacing one occurrence of x_{n-1} by x_n.

If $m \neq x_1^k$, then these words are distinct, so that m is covered by exactly $2 + \sum_{i=1}^{n-1} a_i$ different words.

The cover relation, cont

The following words are covered by m:

- $x_{i_2} \cdots x_{i_d}$, if $i_1 = 1$,
- $x_{i_1} \cdots x_{i_{d-1}}$, if $i_d = 1$,
- The a_2 words obtained by replacing on occurrence of x_2 with x_1, and so on, up to and including the a_N words obtained by replacing one occurrence of x_N by x_{N-1}.

If $m \neq x_1^k$, then these words are distinct, so that m covers exactly $b + \sum_{i=2}^{n} a_i$ different words, where b is the the total number of x_1's in the first and last position together. x_1^k covers exactly 1 word, namely x_1^{k-1}.

Sorted term orders

A standard term order $\geq$ is sorted if whenever $i \leq j$, $s, t \in X^*$ then $t x_i x_j s \geq t x_j x_i s$. $Q =$ the intersection of all sorted term orders.

m is bigger than m' w.r.t Q if m can be obtained from m' by repeated applications of

1. $u \mapsto sut$, $s, t \in X^*$,
2. $t x_i s \mapsto t x_j s$, $i < j$,
3. $t x_j x_i s \mapsto t x_i x_j s$, $i < j$

Galois co-connection

We define an "inverse" to $\sigma : X^* \to [x_1, \ldots, x_n]$.

$$\begin{aligned} [x_1, \ldots, x_n] &\to X^* \\ x_1^{a_1} \cdots x_n^{a_n} &\mapsto x_1^{a_1} \cdots x_n^{a_n} \end{aligned} \tag{10}$$

Order $[x_1, \ldots, x_n]$ with $\mathfrak{C}$, and X^* with Q. Then

1. σ and σ^+ are order-preserving,
2. $\sigma(\sigma^+(m)) \geq m$ for all $m \in [x_1, \ldots, x_n]$,
3. $\sigma^+(\sigma(t)) \geq t$ for all $t \in X^*$,

This Galois coconnection relates the Möbius function of Q to that of the Young lattice.

Questions

1. What can be said about the incidence algebra of $\mathfrak{N}$? In particular, about its Möbius function? (σ, σ^{+}) are not a Galois-coconnection between $\mathfrak{N}$ and $\mathfrak{C}$.
2. Enumerative results about filters in $\mathfrak{N}$ restricted to the set of words of some given total degree?
3. Asymptotics of number of different standard term orders “up to a given total degree”?
4. Non-commutative sand-pile models???