# A graded subring of an inverse limit of polynomial rings

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**Abstract**

We study the power series ring $R = K[[x_1, x_2, x_3, \dots]]$ on countably infinitely many variables, over a field $K$, and two particular $K$-subalgebras of it: the ring $\tilde{R}$, which is isomorphic to an inverse limit $\varprojlim_{n \in \mathbb{N}} K[x_1, \dots, x_n]$ of the polynomial rings in finitely many variables over $K$, and the ring $R'$, which is defined as follows: denote by $R_d \subset R$ the subset consisting of homogeneous power series of total degree $d$; then $R = \prod_{d \in \mathbb{N}} R_d$, whereas $R' = \coprod_{d \in \mathbb{N}} R_d$.

Of particular interest are the homogeneous, finitely generated ideals in $R'$, among them the *generic ideals*. The definition of $\tilde{R}$ as an inverse limit yields a set of *truncation homomorphisms* $\rho_n : \tilde{R} \to K[x_1, \dots, x_n]$ which restrict to $R'$: we have that for $I \subset R'$ generic, $\rho_n(I) \subset K[x_1, \dots, x_n]$ is a generic ideal in the usual sense. It is shown in **Initial ideals of Truncated Homogeneous Ideals** that the initial ideal of such an ideal converge to the initial ideal of the corresponding ideal in $R'$. This initial ideal need no longer be finitely generated, but it is always *locally finitely generated*: this is proved in **Gröbner Bases in $\mathbf{R'}$**. We show in **Reverse lexicographic initial ideals of generic ideals are finitely generated** that the initial ideal of a generic ideal in $R'$ is finitely generated. This contrast to the lexicographic term order.

If $I \subset R'$ is a homogeneous, locally finitely generated ideal, and if we write the Hilbert series of the truncated algebras $K[x_1, \dots, x_n]/\rho_n(I)$ as $q_n(t)/(1-t)^n$, then we show in **Generalized Hilbert Numerators** that the $q_n$'s converge to a power series in $\mathbb{Z}[[t]]$ which we call the *generalized Hilbert numerator* of the algebra $R'/I$.

In **Gröbner bases for non-homogeneous ideals in $\mathbf{R'}$** we show that the calculations of Gröbner bases and initial ideals in $R'$ can be done also for some non-homogeneous ideals, namely those which have an *associated homogeneous ideal* which is locally finitely generated.

The fact that $\tilde{R}$ is an inverse limit of polynomial rings, which are naturally endowed with the discrete topology, provides $\tilde{R}$ with a topology which makes it into a complete Hausdorff topological ring. The ring $R'$, with the subspace topology, is dense in $\tilde{R}$, and the latter ring is the Cauchy completion of the former. In **Topological properties of $\mathbf{R'}$** we show that with respect to this topology, locally finitely generated ideals in $R'$ are *closed*.

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CONTENTS

# 0. INTRODUCTION

Teach thy necessity to reason thus:
There is no virtue like necessity.

*William Shakespeare*

The motivation for introducing the non-noetherian, commutative algebras which are studied in this thesis is the following: they provide the proper habitat for “generic forms in infinitely many variables”, and for ideals generated by such creatures. In particular, we are interested in *initial ideals* of these “generic ideals”. The desire to construct and investigate such seemingly esoteric objects, which correspond to monoid ideals in a countably generated, free abelian monoid, is fueled by our ambition to more fully understand their more mundane and well-known brethren: the *initial ideals of generic ideals* in ordinary polynomial rings over a field $K$.

Let us therefore briefly try to summarize what is known about the latter class of (monomial) ideals. We start by recalling some basic facts about polynomial and power series rings, and about generic forms and generic ideals in polynomial rings.

## *0.1 Generic forms and generic ideals*

First of all, if $n$ is a positive integer, then we denote by $\mathcal{M}^n$ the free abelian monoid (semigroup with unit) on the set $\{x_1, \ldots, x_n\}$. A typical element $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called a *monomial*. There is a natural *logarithmic isomorphism* $\mathcal{M}^n \to \mathbb{N}^n$, where $\mathbb{N}^n$ is the direct sum of $n$ copies of the additive monoid of the natural numbers. This isomorphism is given by $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto (\alpha_1, \ldots, \alpha_n)$.

We now form the *monoid ring* $K[x_1, \ldots, x_n]$ as the set of all *finitely supported* maps $\mathcal{M}^n \to K$, where $K$ is a field (which we for simplicity may take to be the field $\mathbb{C}$ of complex numbers). This set is given the structure of a $K$-algebra by pointwise addition and multiplication with scalars, and with multiplication given by the *convolution*

$$fg(m) = \sum_{m'm''=m} f(m')g(m'').$$

If we drop the condition that the maps be finitely supported, we get instead the *power series ring* $K[[x_1, \dots, x_n]]$.

There is a natural function $\mathcal{M}^n \to \mathbb{N}$ which is uniquely defined by demanding that it takes the *variables* $x_1, \dots, x_1$ to 1, and that it should be a monoid homomorphism. We call the value of this function on a monomial $m \in \mathcal{M}^n$ the *total degree* of the monomial, and denote it by $|m|$. Denoting by $\mathcal{M}^n_d$ the subset of all monomials in $\mathcal{M}^n$ of total degree $d$, we can write $\mathcal{M}^n = \cup_{d=0}^{\infty} \mathcal{M}^n_d$.

We say that a power series $f \in K[[x_1, \dots, x_n]]$ is *homogeneous of degree* $d$ if all monomials in its support $\mathrm{Supp}(f) \subset \mathcal{M}^n$ have total degree $d$. Then, denoting by $K[[x_1, \dots, x_n]]_d$ the subset of all $d$-homogeneous elements, we have that

$$K[[x_1, \dots, x_n]] = \prod_{d=0}^{\infty} K[[x_1, \dots, x_n]]_d,$$

whereas

$$K[x_1, \dots, x_n] = \coprod_{d=0}^{\infty} K[x_1, \dots, x_n]_d.$$

We see that $K[x_1, \dots, x_n]$ is an $\mathbb{N}$-graded ring; we call the homogeneous elements of degree $d$ *forms* of degree $d$.

For a form $f \in K[x_1, \dots, x_n]$ of degree $d$ to be a *generic form*, it should fulfill some additional premises:

(i) $\mathrm{Supp}(f) = \mathcal{M}^n_d$,

(ii) The restriction $f : \mathcal{M}^n_d \to K$ should be injective.

(iii) The set of coefficients of $f$ should be algebraically independent over the prime field of $K$.

A typical element in $K[x_1, \dots, x_n]$ can be written $\sum_{m \in \mathcal{M}^n} c_m m$, where $c_m = f(m) = \mathrm{Coeff}(m, f)$ are elements in $K$, almost all zero. We will henceforth prefer the notation $\mathrm{Coeff}(\cdot, f) : \mathcal{M}^n \to K$ to $f : \mathcal{M}^n \to K$, so as not to confuse the expression $f(m)$ with the *evaluation homomorphism* $K^n \to K$ that is naturally associated to $f$.

For a form of degree $d$, $c_m$ should be zero whenever $|m| \neq 0$. If in addition the form is generic, then for all $m \in \mathcal{M}^n_d$, we must have that $c_m$ is non-zero and not an element of the prime field.

A generic ideal $I \subset K[x_1, \dots, x_n]$ is an ideal generated by finitely many generic forms, which have disjoint sets of coefficients, and for which the union of their sets of coefficients is algebraically independent over the prime field of $K$. Geometrically, if $I = (f_1, \dots, f_r)$ then the zero-set $V(I)$ in affine $n$-space is the intersection of $r$ "generic hypersurfaces".

## 0.2 Gröbner bases and initial ideals

### 0.2.1 Term orders

We recall that any abelian, cancellative, torsion-free monoid admits a total order compatible with its monoid operation [36, Corollary 3.4]. In particular, the free abelian monoid $\mathcal{M}^n$ admits a total order $>$ compatible with the monoid operation; in fact, it has infinitely many such total orders, which we call *term orders*. These total orders were studied already by Macaulay [52], and classified by Robbiano [70, 71] (see also [1, Chapter II, section 8]). Further references on term orderings are [9, 11, 25].

Three of the most commonly used term orders are the *lexicographic*, *graded lexicographic* and *graded reverse lexicographic* orders. On $\mathbb{N}^n$, they are defined as follows. For the lexicographic order,

$$(\alpha_1, \ldots, \alpha_n) >_{\mathrm{lex}} (\beta_1, \ldots, \beta_n)$$

iff the first non-zero component of $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is positive. For the graded lexicographic order,

$$(\alpha_1, \ldots, \alpha_n) >_{\mathrm{glex}} (\beta_1, \ldots, \beta_n)$$

iff $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$ or if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and

$$(\alpha_1, \ldots, \alpha_n) >_{\mathrm{lex}} (\beta_1, \ldots, \beta_n).$$

For the graded reverse lexicographic order,

$$(\alpha_1, \ldots, \alpha_n) >_{\mathrm{rlex}} (\beta_1, \ldots, \beta_n)$$

iff $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$ or if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the last non-zero component of $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is negative.

### 0.2.2 Leading power products and initial ideals

If $>$ is a term order on $\mathcal{M}^n$, any finite subset $S \subset \mathcal{M}^n$ has a maximal (with respect to $>$) element $m$. In particular, if $S = \mathrm{Supp}(f)$, where $f \in K[x_1, \ldots, x_n]$, the maximal element is called the *leading power product* or *leading monomial* and is denoted $\mathrm{Lpp}(f)$. The set of leading power products of an ideal $I$ constitutes a *monoid ideal* in the monoid $\mathcal{M}^n$. To this monoid ideal, there corresponds naturally, via the inclusion $\mathcal{M}^n \to K[x_1, \ldots, x_n]$, a monomial ideal, which we denote by $\mathrm{gr}_>(I) = \mathrm{gr}(I)$ and call the *initial ideal* of $I$ (with respect to $>$).

### 0.2.3 Gröbner bases

A (finite) subset $F \subset I$ which has the property that

$$\text{in}(F) = \{\,\text{Lpp}(f) | f \in F\}$$

generates $\text{gr}(I)$ is called a *Gröbner basis* for $I$. Any Gröbner basis is a generating set of the ideal, but the converse is not true. This is similar to the fact, noted by Macaulay [51] that any in-homogeneous ideal in $K[x_1, \ldots, x_n]$ has an $H$-*basis* which may contain more elements than is necessary for a generating set. The theory of H-bases, associated homogeneous ideals, homogenization, and syzygy computation is an important part of 20'th century mathematics [40, 41, 42, 61, 62, 63].

With the aid of the division algorithm for the Euclidian domain $K[x]$, we can, given any $h, p \in K[x]$, express $h$ as a sum $h = qp + \tilde{h}$, where $\tilde{h}$ is either zero, or the leading power product (that is, the term of highest total degree) of $r$ is strictly smaller than that of $h$. We can generalize this to $K[x_1, \ldots, x_n]$, and for $h \in K[x_1, \ldots, x_n]$, $F = \{f_1, \ldots, f_r\} \subset K[x_1, \ldots, x_n]$ write

$$h = \sum_{i=1}^{r} q_i f_i + \tilde{h}, \quad \text{Lpp}(q_i f_i) \leq \text{Lpp}(h) \text{ and}$$
$$\tilde{h} = 0 \text{ or } \text{Mon}(\tilde{h}) \cap \langle \text{in}(F) \rangle = \emptyset.$$

We say that $\sum_{i=1}^{r} q_i f_i$ is an *admissible combination* of elements in $F$, and that $\tilde{h}$ is an *normal form* of $h$ with respect to $F$.

### 0.2.4 The Buchberger algorithm

[1] There is a $\mathcal{M}^n$-multihomogeneous exact sequence

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{r} K[x_1, \ldots, x_n] E_i \to (\text{Lpp}(f_1), \ldots, \text{Lpp}(f_r)) \to 0$$

where the non-trivial map is given by $E_i \mapsto \text{Lpp}(f_i)$, and where $E_i$ is a formal variable that is given the appropriate $\mathcal{M}^n$-multidegree (that of $\text{Lpp}(f_i)$) so that this map preserves multidegrees. There is also a presentation

$$\eta : \bigoplus_{i=1}^{r} K[x_1, \ldots, x_n] E_i \to (f_1, \ldots, f_r)$$
$$E_i \mapsto f_i$$

[1] This presentation is an adaption of a section in [2].

An essential criteria for a set

$$F\{f_1, \dots, f_r\} \subset K[x_1, \dots, x_n]$$

of monic polynomials to be a Gröbner basis of the ideal $I$ that it generates is the following: it is necessary and sufficient that for each $u \in \mathcal{K}$, the element $\eta(u) \in I$ can be expressed as an admissible combination of elements in $F$. It is easy to see that the syzygy module $\mathcal{K}$ can be generated by pairs, that is, by elements of the form

$$z_{ij} = m_j E_i - m_i E_j,$$

where $m_i, m_j \in \mathcal{M}^n$ has multidegrees so that $z_{ij}$ becomes multihomogeneous of multidegree lcm$(\mathrm{Lpp}(f_i), \mathrm{Lpp}(f_j))$. We can therefore express this condition as follows: all *S-polynomials* (the elements $\eta(z_{ij})$) must *reduce to zero with respect to* $F$ (that is, be admissible combinations of elements in $F$). This is the theorethical motivation for the *Buchberger algorithm* for calculating a Gröbner basis for the ideal generated by a finite set of polynomials.

We shall not dwell longer on this subject; there are several good texts on the subject, to which we refer the reader. Buchbergers papers on the subject are [18, 19, 22, 20, 21]. More recent introductory expositions are [11, 72, 59, 28].

## 0.3 Initial ideals of generic ideals

It is a well-known fact (for a proof, see for instance [81]) that although there exists infinitely many term orders on $\mathcal{M}^n$, if we fix an ideal $I \subset K[x_1, \dots, x_n]$ and partition the term orders into equivalence classes, two term orders $>$ and $>'$ being considered as equivalent if $\mathrm{gr}_>(I) = \mathrm{gr}_{>'}(I)$, then there are only finitely many such equivalence classes. Furthermore, each such equivalence class contains an *archimedian* term order, which corresponds via Robbiano's classification [70] to a single vector in $\mathbb{R}^n$. This is treated in detail in [54, 8, 81].

Conversely, if we fix positive integers $r, d_1, \dots, d_r$, with $r \leq n$, and consider all homogeneous ideals generated by generators

$$f_1, \dots, f_r \subset K[x_1, \dots, x_n]$$

which are homogeneous of degree $d_1, \dots, d_r$, then this set of ideals is in a natural way an affine algebraic set, parameterized by the coefficients of the generators, hence of dimension $N = \sum_{i=1}^{r} \binom{n+d_i-1}{d_i-1}$. It is shown in [31, 29] that if we partition this affine set into equivalence classes, identifying points that corresponds to ideals with identical *Hilbert series*, then there are only finitely many such equivalence classes. Furthermore, there is one component where the Hilbert series (of the quotient $K[x_1, \dots, x_n]/I$) is $(1-t)^{-n}\prod_{i=1}^{r}(1-t)$, and this component contains a Zariski-open set. The generic ideals are contained in this component.

It is well known that the initial ideal $\text{gr}(I)$ of a homogeneous ideal has the same Hilbert series as that of the ideal itself. Since there are only finitely many monomial ideals in $K[x_1, \dots, x_n]$ with a given Hilbert series, we conclude that if we fix a term-order $>$, and partition $\mathbb{A}_K^N$ into equivalence classes, identifying two points if the ideal that they represent have the same initial ideal with respect to $>$ (this is a refinement of the partition of $\mathbb{A}_K^N$ described in the previous paragraph) then there is only finitely many equivalence classes, and one of the components contains a Zariski-open set. Once again, the generic ideals are contained in this "big" component. This is proved in a different way by Weispfennig [85], using so-called *comprehensive Gröbner bases*. Basically, he shows that the initial ideal of an ideal in the class under consideration (viewed as a point in $\mathbb{A}_K^N$) is determined, once we know, for a finite number of polynomials in $N$ variables, for which of these polynomials the point is a root. One can construct a binary tree which encodes this data, and for which the leaves are all initial ideals.

Before we conclude this section, which we hope has convinced the reader that the initial ideals of generic ideals are important and natural objects to study, we mention briefly the notion of the *generic initial ideal*, $\text{gin}(I)$, of an homogeneous ideal $I \subset K[x_1, \dots, x_n]$, with respect to a term order $>$. If $g = (g_{ij})$ is an element of the general linear group of the $K$-vector space $K[x_1, \dots, x_n]_1$, that is, the vector space spanned by the variables, then $g$ acts in a natural way on $K[x_1, \dots, x_n]$. Hence, $g$ transforms the homogeneous ideal $I$ to another homogeneous ideal $g(I)$, which may be viewed as the original ideal, expressed in other coordinates. Galligo proved in 1974 [32] that there is a Zariski-open set of invertible transformations $g$ for which $\text{gr}(g(I))$ is constant. This constant value is denoted $\text{gin}(I)$, and called the *generic initial ideal* of $I$. We refer to [39] and a section in [25] for a more information on gin. Here, we shall only note that for a generic ideal $I$, $\text{gin}(I) = \text{gr}(I)$, and that the initial ideals of generic ideals therefore has the following property, which is common to all generic initial ideals: it is *strongly stable* or *Borel*. A monomial ideal is Borel if it is stable under the action of the Borel subgroup (the group of all upper-triangular matrices) of the general linear group on $K[x_1, \dots, x_n]_1$. An equivalent formulation is this: whenever a monomial $m$ is in the monomial ideal, and $m$ is divisible by $x_i$, then $mx_j/x_i$ is in the monomial ideal for all $1 \leq j < i$. The transformation $m \mapsto mx_j/x_i$ is called an *elementary move*. For more information on Borel-stable monomial ideals, we refer to [32, 12, 13].

## *0.4 Degrevlex initial ideals of generic ideals*

Suppose that we are to calculate a Gröbner basis for an ideal I in the polynomial ring $K[x_1, \dots, x_n]$, and that we are free to choose any term order that we desire. Often the the graded reverse lexicographic term order will yield the smallest Gröbner basis, in terms of the maximal total degrees of the generators, and in terms of the number of generators in the Gröbner basis. It is therefore a natural first step, when embarking on the ambitious project of determining the initial ideals of generic ideals, to first try to deal with the graded reverse lexicographic (degrevlex, for short) term order.

A key property of the degrevlex term order is that it "commutes with truncation homomorphisms". Let $r < n$ be a positive integer, and denote by $\rho_r$ the composite

$$K[x_1, \dots, x_n] \twoheadrightarrow \frac{K[x_1, \dots, x_n]}{(x_{r+1}, \dots, x_n)} \simeq K[x_1, \dots, x_r].$$

Then, for any term order $>$ we have that $\rho_r(\mathrm{gr}(I)) \subset \mathrm{gr}(\rho_r(I))$, but in general the reverse inclusion does not hold. For the degrevlex term order the reverse inclusion *does* hold, which has as a consequence that if I is a generic ideal (in $K[x_1, \dots, x_n]$) generated by $r$ generic forms, then $\mathrm{gr}(I)$ is generated in $K[x_1, \dots, x_r]$ (a simple proof of this well-know fact is given in Lemma 3.3.2). Furthermore, Moreno proves in his thesis [55] that if $m, m'$ are two minimal monomial generators of $\mathrm{gr}(I)$ (that is, they are minimal generators of the corresponding semigroup ideal) then if $x_n^v | m$ and $|m'| > |m|$, then $x_n^v | m'$.

It is widely believed that the degrevlex initial ideal of generic ideals (not just the complete intersection case when we have no more generators than variables) is determined by the Hilbert series of $K[x_1, \dots, x_n]/I$. Before describing what the conjectured structure of the inital ideal is, we mention that the Hilbert series of $K[x_1, \dots, x_n]/I$ is conjectured to be (see [27, 30, 45])

$$\left\lceil \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1 - t)^n} \right\rceil,$$

where $I = (f_1, \dots, f_r)$ is a generic ideal with $|f_i| = d_1$, but where we might have that $r > n$, and where $\lceil \sum a_i t^i \rceil = \sum b_i t^i$ with $b_i = a_i$ if $a_j > 0$ for all $j \leq i$, and $b_i = 0$ otherwise.

Continuing with the description of the conjectured structure of the initial ideal: from the Hilbert series, and from the minimal monomial generators $m_1, \dots, m_k$ of degree $< d$ of $\mathrm{gr}(I)$, the minimal monomial generators of $\mathrm{gr}(I)$ of degree $d$ are given as follows, according to the conjecture: take the first $s$ monomials in $\mathcal{M}_d^n \setminus (m_1, \dots, m_k)$, where $s$ is given by the difference of the coefficients of the $t^d$ term of the Hilbert series of the monomial ideal $(m_1, \dots, m_k)$ and of the corresponding coefficient of the known Hilbert series of the ideal $I$. See Section 3.3.1 for an example. This conjecture has been checked by computer for a very large number of cases, and the computational "evidence" for its veracity is overwhelming. The special cases $r = 2$ or $n = 2$ are easy to analyze. It turns out that in these cases, the conjecture can be proved easily.

Namely, if we prove it for $n = 2$, then we know that for the case of $r = 2$, the initial ideal is generated in $K[x_1, x_2]$, and coincides with the initial ideal of the corresponding generic ideal in $K[x_1, x_2]$. On $\mathcal{M}^2$, there is only one term order which refines the partial order given by the total degree relation, and which fulfills $x_1 > x_2$. The fact that the initial ideal of a generic ideal must be Borel-fixed can be expressed as follows: the set of monomials generators of the initial ideal must, for any degree $d$, be an up-set (or filter, see [83] for definitions) with respect to the partial order on monomials of degree $d$ given by the *strongly stable relation*: $m \geq m'$ iff $m'$ can be transformed into $m$ by a sequence of elementary moves. Now, for $n = 2$, this partial order is a total order, and must therefore coincide with degrevlex. Thus, the initial ideal is in each total degree $d$ generated by an up-set in $\mathcal{M}^2_d$ with respect to the degrevlex total order. This is what the conjecture claims.

## 0.5 Lexicographic initial ideals of generic ideals

It seems natural to assume that the initial ideals of generic ideals, with respect to other term orders, would fulfill the same property as the one conjectured to hold for the degrevlex term order. However, this is far from the truth. Already for the lexicographic term order (or the graded lexicographic term order: since generic ideals are homogeneous, these two term orders yield the same initial ideal) there is a plethora of counterexamples. The simplest is the case of the generic ideal generated by two generic quadratic forms[2].

We are studying the generic ideal $I \subset K[x_1, \ldots, x_n]$, generated by the two quadratic generic forms

$$\begin{aligned} f_1 &= \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} x_i x_j \\ f_2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} x_i x_j. \end{aligned} \tag{0.1}$$

[2] This counterexample was communicated to Ralf Fröberg by Bernd Sturmfels and David Eisenbud on a conference in 1993. Ralf then suggested it to me as an interesting problem to study.

For $n \geq 2$, the Hilbert series of $K[x_1, \ldots, x_n]/I$ is $(1-t^2)^2/(1-t)^n$. The degrevlex initial ideal of I is $(x_1^2, x_1x_2, x_2^3)$. This agrees with what we proved above, since this monomial ideal has the correct Hilbert series, and each generator is the "first available one" with respect to degrevlex. We say that there are no "holes" in the monomial ideal.

For the lexicographic term order (or the graded lexicographic term order: the ideals in question are homogeneous, so these two term orders yield the same initial ideals) the initial ideal is once more $(x_1^2, x_1x_2, x_2^3)$ when $n = 2$. This comes as no surprise, since degrevlex and lex coincides on $\mathcal{M}^2$. When $n > 2$ however, the lexicographic initial ideal becomes $(x_1^2, x_1x_2, x_1x_3^2, x_2^4)$. Here, we observe that $x_1x_3^2$ is the lexicographically first of the available monomials of total degree 3, but that the lexicographically first of the available monomials of total degree 4 is $x_1x_3x_4^2$. For large $n$, the "hole" in $\mathcal{M}_4^n$ between this monomial and $x_2^4$ is very large. We see that the structure of the lexicographic initial ideals of generic ideals are governed by other, more complicated rules than those that determine the degrevlex initial ideals of generic ideals.

It is even more instructive to study the case of a generic ideal generated by a quadratic and a cubic form[3]. The degrevlex initial ideal, for $n \geq 2$, is $(x_1^2, x_1x_2^2, x_2^4)$. The lexicographic initial ideals are as shown in Table 0.1 on page xvii.

| $n$ | lex-initial ideal of $I \subset K[x_1, \ldots, x_n]$ |
|---|---|
| 2 | $(x_1^2, x_1x_2^2, x_2^4)$ |
| 3 | $(x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_3^4, x_2^6)$ |
| 4 | $(x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, x_1x_2x_4^4, x_2^6, x_1x_3^6)$ |
| 5 | $(x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, x_1x_2x_3x_4x_5^2, x_2^6, x_1x_2x_3x_5^4, x_1x_3^6, x_1x_2x_4^6)$ |
| 6 | $(x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, x_1x_2x_3x_4x_5^2, x_2^6, x_1x_2x_3x_4x_5x_6^2,$ $x_1x_3^6, x_1x_2x_3x_4x_6^4, x_1x_2x_4^6, x_1x_2x_3x_5^6)$ |
| 7 | $(x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, x_1x_2x_3x_4x_5^2, x_2^6, x_1x_2x_3x_4x_5x_6^2,$ $x_1x_3^6, x_1x_2x_3x_4x_5x_6x_7^2, x_1x_2x_4^6, x_1x_2x_3x_4x_5x_7^4, x_1x_2x_3x_5^6, x_1x_2x_3x_4x_6^6)$ |

***Tab. 0.1:*** *The lex-initial ideal of a generic ideal generated by a quadratic and a cubic form*

As $n$ increases, the lex-initial ideal requires ever more generators. There is no $N$ such that for $n \geq N$ the initial ideals stabilize, in contrast to the case of the degrevlex term order. However, if we *fix a total degree* $d$, and concentrate

[3] This case was studied by Alyson Reeves [60].

on the monomial generators of such degree, then as $n$ varies, we note that these monomials do in fact seem to stabilize. In fact, computer calculations seems to indicate that for large enough $n$ (depending on $d$) the monomial generators of degree $d$, for $d \geq 6$, are $x_1 x_2 x_3 \cdots x_{d-2} x_{d_1}^2$ and $x_1 x_2 x_3 \cdots x_{d-6} x_{d-4}^6$.

It is not at all unnatural to ask the question: can the totality of "stable" monomial generators be regarded as the generators of some monomial ideal in some over-ring of all polynomial rings $K[x_1, \ldots, x_n]$? Can it in fact be the *initial ideal* of some ideal in this (by necessity non-noetherian) ring? And if so, is this latter ideal perhaps to be regarded as a *generic ideal* in this phantasmagorical ring that we envision, and is it generated by elements that may boldly be christened as *generic forms in infinitely many variables*? To answer these questions affirmatively is what the first two articles of this thesis sets out to do. We will in the next sections sketch briefly how this is done.

## 0.6 The power series ring on infinitely many variables, and some of its subrings

At this point, we are looking for some ring that

1. Contains the polynomial ring $K[x_1, \ldots, x_n]$ as a subring, for all integers $n$,
2. Allows a multitude of term-orders $>$ such that every element in the ring has a leading power product with respect to $>$,
3. Is non-noetherian, and furthermore has the property that a finitely generated, homogeneous ideal might have a non-finitely generated initial ideal.

The reason for the last requirement is the example just studied, that of the initial ideal of a generic ideal generated by a quadratic and cubic form, which we believe should have an initial ideal which needs infinitely many generators, two for each sufficiently large total degree.

We form the free abelian monoid $\mathcal{M}$, generated by the countable set $\{x_1, x_2, x_3, \ldots\}$ of variables. Then, $\mathcal{M} = \varinjlim \mathcal{M}^n$. There is an surjective monoid homomorphism

$$\rho_n : \mathcal{M} \longrightarrow \mathcal{M}^n \cup \{0\}$$

$$\rho_n(m) = \begin{cases} m & \text{if } m \in \mathcal{M}^n \\ 0 & \text{otherwise.} \end{cases}$$

The inclusion $\mathcal{M}^n \hookrightarrow \mathcal{M}$ is "almost a section" to this map, which we call the $n$*'th truncation homomorphism.*

Proceeding along the path trodden in Section 0.1, we define

$$R = K[[x_1, x_2, x_3, \ldots]]$$

as the set of all maps $\mathcal{M} \to K$, and $K[x_1, x_2, x_3, \ldots]$ as the subring of all finitely supported maps. Both these rings contains all polynomial rings. The ring $K[x_1, x_2, x_3, \ldots]$ allows the definition of leading power products, but the ring $R$ does not, since the set $\left\{ x_1^i \middle| i \in \mathbb{N} \right\} \subset \mathcal{M}$ can not have a maximal element with respect to to a term order $>$ on $\mathcal{M}$. The ring $K[x_1, x_2, x_3, \ldots]$ fails to satisfy the last requirement: any finitely generated ideal $I \subset K[x_1, x_2, x_3, \ldots]$ is generated in some $K[x_1, \ldots, x_n]$, and the initial ideal is likewise generated in $K[x_1, \ldots, x_n]$, hence is finitely generated.

It is now clear that our elusive ring, if it exists, must be a ring strictly containing $K[x_1, x_2, x_3, \ldots]$ and strictly contained in $R$. Noting that

$$K[x_1, x_2, x_3, \ldots] = \varinjlim K[x_1, \ldots, x_n],$$

it is natural to try instead the *projective limit* $\tilde{R} = \varprojlim K[x_1, \ldots, x_n]$. We need to define the connecting homomorphisms

$$K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_{n-1}].$$

These are given by the maps

$$K[x_1, \ldots, x_n] \to \frac{K[x_1, \ldots, x_n]}{(x_n)} \simeq K[x_1, \ldots, x_{n-1}].$$

Noting that the truncation homomorphism $\mathcal{M} \to \mathcal{M}^n \cup \{0\}$ can be linearly extended to a map $R \to K[[x_1, \ldots, x_n]]$ by the formula

$$\rho_n \left( \sum_{m \in \mathcal{M}} c_m m \right) = \sum_{m \in \mathcal{M}} c_m \rho_n(m) = \sum_{m \in \mathcal{M}^n} c_m m$$

and that there are maps

$$K[[x_1, \ldots, x_n]] \to \frac{K[[x_1, \ldots, x_n]]}{(x_n)} \simeq K[[x_1, \ldots, x_{n-1}]],$$

it is easy to see that $R = \varprojlim K[[x_1, \ldots, x_n]]$, and that

$$\tilde{R} \simeq \{ f \in R | \forall n : \rho_n(f) \in K[x_1, \ldots, x_n] \}.$$

As an aside, we mention that the construction of $\varprojlim K[x_1, \ldots, x_n]$ can be made a bit more generally by choosing a set $X$ of variables and then forming the inverse limit of all polynomial rings with variables given by a finite subset of $X$.

This ring has been studied by combinatorians [15] under the name of *the ring of formal polynomials*. If the set $X$ is countable, then this construction is no more general than ours, since every denumerable directed partially ordered set contains a denumerable cofinal chain [26].

The ring $\tilde{R}$ will play a role in our further investigations, but it is not the ring that we at this moment seek. It does not allow the definition of leading power products: the element

$$x_1 + x_1x_2 + x_1x_2x_3 + x_1x_2x_3x_4 + \cdots \in \tilde{R}$$

can have no leading power product.

It is time to reveal the structure of the "correct" ring for the purpose of this investigation. First, we remark that may, just as in Section 0.1, say that an element $f \in R$ is *homogeneous of degree* $d$ if all monomials in its support $\mathrm{Supp}(f) \subset \mathcal{M}$ have total degree $d$. This does certainly not mean that $R$ is graded, but we can at least write $R = \prod_{d\in\mathbb{N}} R_d$. Now define $R' = \coprod_{d\in\mathbb{N}} R_d$. This is a graded subring of $R$ and of $\tilde{R}$, and it contains $K[x_1, x_2, x_3, \ldots]$ as a graded subring. Furthermore, it allows the definition of leading power products with respect to arbitrary term orders, as is shown in **Gröbner bases in $\mathbf{R'}$**, the first article of this thesis. In this article, a somewhat artificial condition on the term order $>$ on $\mathcal{M}$ is added: it should order the variables as $x_1 > x_2 > x_3 > \ldots$. It is evident that we need to have some sort of condition on $>$ restricted to the degree 1 elements in $\mathcal{M}$, since this restriction must certainly be the inverse relation of a well order in order for linear forms to have leading power products. This latter condition is shown to be sufficient in the appendix to **Reverse lexicographic initial ideals of generic ideals are finitely generated**.

Finally, in $R'$ it might happen that the initial ideal of an homogeneous, finitely generated ideal $I$ is not finitely generated. It is however always *locally finitely generated*, which means that it can be generated by a countable set, containing but finitely many elements of any given total degree. For instance, the lex-initial ideal of a generic ideal generated by a quadratic and a cubic *generalized generic form* in $R'$ most likely has 2 generators of any given total degree $d$, namely $x_1x_2x_3\cdots x_{d-2}x_{d_1}^2$ and $x_1x_2x_3\cdots x_{d-6}x_{d-4}^6$.

## 0.7 Gröbner bases in $R'$

The calculation of initial ideals in $R'$ is best performed by constructing a *Gröbner basis* of the ideal under consideration, starting by a set of generators. For this purpose, one uses a modified version of the so-called *Buchberger algorithm*, adding *normal forms* of *S-polynomials*. Since $R'$ is non-noetherian, this process need not stop, however, we can calculate a *partial Gröbner Basis* up to degree $d$ using only a finite number of operations in $R'$. In particular, this partial basis will be finite. This allows us to solve the *membership problem* for finitely generated homogeneous ideals in $R'$.

Let us as an example take the calculation of the degrevlex-initial ideal of the generic ideal generated by two generic quadratic forms in $R'$. Here

$$\begin{aligned} f &= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \alpha_{ij} x_i x_j \\ g &= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \beta_{ij} x_i x_j \end{aligned} \tag{0.2}$$

(compare with (0.1)) and $I = (f, g)$. A Gröbner basis for I is given by $f, g$ and

$$h = x_2 f - x_1 g + f \sum_{j=3}^{\infty} \beta_{1j} x_j + g \sum_{j=3}^{\infty} (\beta_{2j} - \alpha_{1j}) x_j.$$

The initial ideal is $(x_1^2, x_1x_2, x_2^3)$.

A corresponding calculation for the lexicographic term order is given at the end of **Gröbner bases in R$'$**.

### *0.7.1 Normal forms in* $R'$

There are two obstacles that has to be overcome, in order for the Gröbner bases theory here sketched to work. First, we must have *normal form (or division) algorithm* in $R'$. It will suffice if we can find the normal form of an element with respect to a finite number of other elements. Secondly, we must show that the so-called Buchberger Criterion holds in $R'$: that is, if all S-polynomials of a set of generators of an ideal reduce to zero with respect to that set, then the set of generators is a Gröbner basis. We will not discuss the second, most delicate condition in this introduction, but we shall have something to say about normal forms.

The crucial result in this area is Proposition 1.3.2:

**Proposition 1.3.2** *Let* $F := \{f_1, \ldots, f_r\} \subset R'$ *consist of monic elements. For* $h \in R'$ *there are* $h_1, \ldots, h_r, \tilde{h} \in R'$ *such that*

$$h = \sum_{i=1}^{r} h_i f_i + \tilde{h}, \quad \mathrm{Lpp}(h_i f_i) \leq \mathrm{Lpp}(h) \textit{ and}$$

$$\tilde{h} = 0 \textit{ or } \mathrm{Mon}(\tilde{h}) \cap \langle \mathrm{in}(F) \rangle = \emptyset.$$

*We say that* $\tilde{h}$ *is a normal form of* $h$ *with respect to* $F$ *and* $>$.

The method used in the proof is the following: since there are only finitely many elements with respect to which the normal form is to be calculated, there are only finitely many leading power products of these elements. Consequently, there is an $N$ such that all these power products are contained in $K[x_1, \ldots, x_N]$. There is an natural isomorphism $R' \simeq C_N[x_1, \ldots, x_N]$, where

$$C_N = K[[x_{N+1}, x_{N+2}, x_{N+3}, \ldots]] \cap R'$$

is a domain of coefficients. Thus, the $h_i \in C_N[x_1, \ldots, x_N]$ can be regarded as *polynomials*, yielding a *reduction system*

$$\begin{bmatrix} \mathrm{Lpp}(h_1) & \mapsto & -h_1 + \mathrm{Lpp}(h_1) \\ \mathrm{Lpp}(h_2) & \mapsto & -h_2 + \mathrm{Lpp}(h_2) \\ \mathrm{Lpp}(h_3) & \mapsto & -h_3 + \mathrm{Lpp}(h_3) \\ & \vdots & \end{bmatrix}$$

Even $h$ can be regarded as an element in $C_N[x_1, \ldots, x_N]$, and furthermore, any monomial in $h$ that is changed by the isomorphisms (because some of its variables are regarded as coefficients) is only altered in variables with index higher than $N$. This does not affect divisibility with $\mathrm{Lpp}(h_i)$, since these monomials are products of variables $x_1, \ldots, x_N$. So, applying the usual division algorithm for $C_N[x_1, \ldots, x_N]$ (there is such an algorithm for polynomial rings with coefficients in a commutative domain) we get a normal form which, when regarded as an element in $R'$, has no monomial which is divisible by some $\mathrm{Lpp}(h_i)$.

This method of "regarding variables as coefficients", performing normal form calculations in the polynomial ring $C_N[x_1, \ldots, x_N]$, then "lifting" the normal form back to $R'$ is simple and convenient, but undoubtedly slightly artificial. More natural, perhaps, would be to perform all calculations inside $R'$, using simple reduction steps of the form

$$m\,\mathrm{Lpp}(h_i) \mapsto m(-h_i + \mathrm{Lpp}(h_i)).$$

For instance, to reduce $\sum_{i=1}^{\infty} x_1 x_i$ with respect to $\{x_1\}$ we would do

$$\sum_{i=1}^{\infty} x_1 x_i \mapsto \sum_{i=2}^{\infty} x_1 x_i \mapsto \sum_{i=3}^{\infty} x_1 x_i \mapsto \sum_{i=4}^{\infty} x_1 x_i \cdots \mapsto 0,$$

where the last reduction is some sort of "limit reduction", and where the reduction chain is indexed by the limit ordinal $\omega$. If we add the reductions, we see that we have performed the reduction

$$\sum_{i=1}^{\infty} x_1 x_i \mapsto (\sum_{i=1}^{\infty} x_1 x_i - x_1 \sum_{i=1} x_i) = 0.$$

Conversely, one can show [75] that the division algorithm described above can be given this "finer structure" of "iterated reductions". Any reduction in $C_N[x_1, \ldots, x_N]$ by means of a homogeneous $h_i$ of degree $d$ then corresponds to an infinite reduction chain of length $\omega^d$. However, the converse does not hold: there are reduction chains that can no be gotten as "lifts" of reductions in $C_N[x_1, \ldots, x_N]$, (not for any $N$), and there are normal forms given by infinite reduction chains which the "regard as coefficients, then lift"-method can not produce. Since this latter method always produces *some* normal forms, it will, for our purposes, suffice.

We remark that the method of reduction chain indexed by large ordinals appears naturally in the study of normal forms in noetherian power series rings [10]. The notion of *standard bases* in these rings is implicit already in the work of Hironaka on the resolution of singularities [44].

### 0.7.2 Locally finitely generated ideals in $R'$

We have already hinted at the fact that a homogeneous, finitely generated ideal $I \subset R'$ *need not* have a finitely generated initial ideal $\mathrm{gr}(I)$. However, from the way that the Buchberger algorithm works for homogeneous indata, and with a *selection strategy* that always chooses the critical pair with lowest total degree, one can prove by induction that there will be but finitely many basis elements of any given total degree. Thus, $\mathrm{gr}(I)$ is what we call a *locally finitely generated* ideal. It is also clear that if we start with a locally finitely generated ideal I, then $\mathrm{gr}(I)$ is also locally finitely generated.

Another way of stating that $\mathrm{gr}(I)$ is locally finitely generated, is to say that for all total degrees $d$,

$$\dim_K \left( \frac{\mathrm{gr}(I)_d}{\sum_{j=1}^{d-1} R'_j \, \mathrm{gr}(I)_{d-j}} \right) < \infty.$$

This formula indicates the proper way of generalizing the above results to certain inhomogeneous ideals $J$: we should consider such $J$ as fulfills

$$\dim_K \left( \frac{\mathcal{T}^{\leq d} J}{\sum_{j=1}^{d-1} \mathcal{T}^{\leq j} R' \mathcal{T}^{\leq d-j} J} \right) < \infty,$$

where $\mathcal{T}^{\leq d} J$ denotes the elements in $J$ of total degree $\leq d$. This is treated in some detail in **Gröbner bases for non-homogeneous ideals in $\mathbf{R}'$**.

## 0.8 Approximating the initial ideal by the initial ideals of its truncations

In Section 0.5, we hinted at the fact the initial ideal of a generic ideal $I$ in $R'$ can be "arbitrarily well approximated" by the initial ideals of the corresponding generic ideals in ordinary polynomial rings $K[x_1, \ldots, x_n]$. By this, we mean that if we fix a total degree $d$, and set out to find the minimal monomial generators of $\mathrm{gr}(I)$ that have total degree $\leq d$, then there is some $N$ such that for $n \geq N$, the minimal monomial generators of the initial ideal of the generic ideal in $K[x_1, \ldots, x_n]$ will coincide with the monomial generators that we seek. If we note that any two generic ideals in $K[x_1, \ldots, x_n]$ of the same type (that is, the same degrees of its generators) have the same initial ideal, and that the truncation $\rho_n(I) \subset K[x_1, \ldots, x_n]$ of a generic ideal in $R'$ is a generic ideal in $K[x_1, \ldots, x_n]$, then it is clear that our claim will follow if we can prove the following result:

$$\forall d : \exists N(d) : \forall n \geq N(d) : \mathcal{T}^{\leq d}\,\mathrm{gr}(\rho_n(I)) \cap \mathcal{M} = \mathcal{T}^{\leq d}\,\mathrm{gr}(I) \cap \mathcal{M}.$$

Equivalently, we want to show that although in general

$$\rho_n(\mathrm{gr}(I)) \neq \mathrm{gr}(\rho_n(I))$$

(except for the degrevlex term order!) we have the weaker condition

$$\mathcal{T}^{\leq d}\rho_n(\mathrm{gr}(I)) = \mathcal{T}^{\leq d}\,\mathrm{gr}(\rho_n(I))$$

when $n \gg d$. As it turns out, we can prove this, not only for generic ideals, but for all locally finitely generated ideals in $R'$. This is done in **Initial ideals of truncated homogeneous ideals**.

One interesting consequence of this is that the initial ideal $\mathrm{gr}(I)$ of an locally finitely generated ideal $I \subset R'$ is completely determined, once we know *all* of its truncated ideals $\rho_n(I)$. In **Topological properties of R′** we show that in addition, $I$ itself is determined by its truncated ideals. This need not be true for non-locally finitely generated ideals. The proper way of expressing this fact turns out to be the following formulation: in the topology defined by the filtration given by the kernels $A_n$ of the truncation homomorphisms $\rho_n : R' \to K[x_1, \ldots, x_n]$, the locally finitely generated ideals are *closed*.

## 0.9 Something about the reverse lexicographic term order

We have mentioned several times already that the degrevlex term order on $\mathcal{M}$ possesses several nice properties, which makes it simpler to analyze. To name but a few of its nice qualities:

1. Initial ideals with respect to degrevlex "commutes" with the truncation homomorphisms.
2. The kernels $A_n$ of the truncation homomorphisms $\rho_n : R' \to K[x_1, \ldots, x_n]$ have the property that $A_n \cap \mathcal{M}_d$ is a degrevlex terminal segment in $\mathcal{M}_d$.
3. If $f \in R'$ is homogeneous, then $f \in A_n \iff \mathrm{Lpp}(f) \in A_n$, if the leading power product is taken with respect to the degrevlex term order.
4. If $h \in R'$ is homogeneous, and if $\nu$ is any positive integer, then either $\rho_\nu(h) = 0$, or $\mathrm{Lpp}(h) = \mathrm{Lpp}(\rho_\nu(h))$.
5. The degrevlex initial ideal of a generic ideal in $K[x_1, \ldots, x_n]$, where the ideal is generated by $r$ forms, is generated in $K[x_1, \ldots, x_r]$.

From the last property it follows immediately that **Reverse lexicographic initial ideals of generic ideals are finitely generated**; the generic ideals are in $R'$, of course. It is an interesting question, if the degrevlex initial ideal of any homogeneous, finitely generated ideal in $R'$ is finitely generated. The article just mentioned does not succeed in answering this question, but does provide some methods that might be of use when investigating this problem.

The first three properties are used in the proof of the closedness of locally finitely generated ideals. In this proof, the key step is the construction of a degrevlex Gröbner basis of the locally finitely generated ideal.

## *0.10 "Hilbert numerators" for locally finitely generated ideals*

One important invariant of a homogeneous ideal $J \subset K[x_1, \ldots, x_n]$ is the *Hilbert series* of the quotient $K[x_1, \ldots, x_n]/J$. If $I \subset R'$ is locally finitely generated, we can not hope to form the Hilbert series of $R'/I$, since each graded piece $R'_d/I_d$ will be an infinite-dimensional $K$-vector space. However, if we study the case when $I$ is a generic ideal, an interesting phenomenon can be observed. Namely, if we perform our standard technique of studying locally finitely generated ideals in $R'$ by means of their truncations, we see that the Hilbert series of $K[x_1, \ldots, x_n]/\rho_n(I)$ is given by

$$\mathrm{Hilb}_{K[x_1,\ldots,x_n]/\rho_n(I)}(t) = \prod_{i=1}^{r} (1 - t^{d_i})/(1-t)^n.$$

Here, we assume that $I = (g_1, \ldots, g_r)$ with $|g_i| = d_i$.

Now, the following simple observation is the key that we need in order to proceed: the polynomial $(1-t)^n \mathrm{Hilb}_{K[x_1,\ldots,x_n]/\rho_n(I)}(t)$ can be regarded as a power series, and as such, it is *constant as* $n$ *increases*, in particular, *it converges to a* *power series as* $n$ *tends to infinity*. As it happens, this is true for all locally finitely generated ideals. It need not be true that this limit, which we call the *Hilbert numerator* of $R'/I$, is a polynomial (it is of course always a polynomial for finitely generated generic ideals). For instance, the locally finitely generated monomial ideal $(x_1, x_2^2, x_3^3, x_4^4, \dots)$ has Hilbert numerator $\prod_{i=1}^{\infty}(1 - t^i)$. This power series is not a polynomial.

It is easy to see that finitely generated monomial ideals have polynomial Hilbert numerators, as have homogeneous ideals generated by two elements. In **Generalized Hilbert Numerators**, we give some additional criteria for when the Hilbert numerator is a polynomial. We do not answer the interesting question: does all finitely generated, homogeneous ideals have polynomial Hilbert numerator? This would follow if for instance all such ideals had a finitely generated degrevlex initial ideal, but it could be true even if this latter statement is wrong.

# 0. ERRATA

This chapter was not included in the printed version of the thesis.

## *0.11 New bibliographic information*

All the articles contained in the thesis have now appeared in print, as


1. Jan Snellman. Gröbner bases and normal forms in a subring of the power series ring on countably many variables. *J. Symbolic Comput.*, 25(3):315–328, 1998.

2. Jan Snellman. Initial ideals of truncated homogeneous ideals. *Comm. Algebra*, 26(3):813–824, 1998.

3. Jan Snellman. Reverse lexicographic initial ideals of generic ideals are finitely generated. In Buchberger and Winkler, editors, *Gröbner Bases and Applications: Proceedings of the Conference 33 years of Gröbner Bases*, volume 251 of *London Mathematical Society Lecture Notes Series*, 1998.

4. Jan Snellman. Generalized Hilbert numerators. *Comm. Algebra*, 27(1):321–333, 1999.

5. Jan Snellman. Non-homogeneous ideals in a graded subring of the power series ring on a countably infinite number of variables over a field. *Int. J. Math. Game Theory Algebra*, 10(5):391–404, 2000.

6. Jan Snellman. Some topological properties of a subring of the power series ring on a countably infinite number of variables over a field. *Int. J. Math. Game Theory Algebra*, 8(4):231–241, 1999.


## *0.12 Detailed errata*

The following errata is relative to the printed version of the thesis, so the line numbers (particularly those refereing to pages in the introduction) may be slightly off, relative to this version.

- page x, 4 lines from top: $x_1, \dots, x_n$.
- page x, 8 lines from bottom: $|m| \neq d$.
- page xii, second display formula: Mon is synonymous to Supp.
- page xii, second line after second display: a normal form
- page xii, third display formula: $\mathcal{K}$ is the kernel.
- page xiii, 4 lines from the bottom: formula should read $N = \sum_{i=1}^{r} \binom{n+d_i-1}{n-1}$.
- page xiv, first line: $(1-t)^{-n} \prod_{i=1}^{r} (1-t^{d_i})$
- page xiv, last paragraph: we assume that the term order $>$ orders the variables as $x_1 > x_2 > \cdots > x_n$.
- page xiv, 10 lines from bottom: for more information
- page xv, line 3 from the top: Often the graded
- page xv, line 9 from bottom: $|f_i| = d_i$.
- page xvi, last display: $f_2 = \sum_{i=1}^{n} \sum_{j=i}^{n} \beta_{ij} x_i x_j$
- page xvii, lines 10-11 from top: the lexicographically first of the available monomials of total degree 4 is $x_1 x_3 x_4^2$ when $n \geq 4$.
- page xviii, second line: $x_1 x_2 x_3 \cdots x_{d-2} x_{d-1}^2$
- page xviii, two lines above second last display: There is a surjective
- page xix, line 5 from top: respect to a term order
- page xx, last line before section 0.7: $x_1 x_2 x_3 \cdots x_{d-2} x_{d-1}^2$
- page xxi: A gröbner basis for I is given by $f', g'$ and $h = x_2 f' - x_1 g' + f' \sum_{j=3}^{\infty} \beta_{1j} x_j + g' \sum_{j=3}^{\infty} (\beta_{2j} - \alpha_{1j}) x_j$, where $f', g'$ are obtained by "Gaussian elimination", that is

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \beta_{11} & \beta_{12} \end{pmatrix}^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} f' \\ g' \end{bmatrix}$$

- page xxii, fifth line: There is a natural isomorphism *of* $K$-*algebras*.

- page 3, second footnote: one should also demand that the set $\{c_m | m \in \mathcal{M}_k^n\}$ has the same cardinality as $\mathcal{M}_k^n$, that is, that all the $c_m$'s are different.

- page 4, second last line in proof of Lemma 1.2.3: Similarly, $p = \prod_{j=1}^{s} w_j$

- page 4, last sentence before Theorem 1.2.4: To see that $R'$ is the largest subalgebra of $R$ with the desired property, let $>$ be any admissible order for which $|m| > |m'| \implies m > m'$, and let $f \in R \setminus R'$. Then $\mathrm{Mon}(f)$ contains power products of arbitrarily high total degree, and hence, $\mathrm{Mon}(f)$ can have no maximal element with respect to $>$.

- page 4, second last line: $j \leq N$

- page 6: Proposition 1.3.2: monic means leading coefficient 1

- page 9, lines 7-9: If $K$ is the binary field, then the assertion is false, since then $\sum_{i=1}^{\infty} x_i^2 = \left(\sum_{j=1}^{\infty} x_j\right)^2$. So we should suppose that $K$ is the field of complex numbers (in fact, it is enough that $K$ has characteristic different from 2). If we denote by $V$ the $K$-vector space of linear forms in $K[x_1, \ldots, x_n]$, then the map

$$\left(S^1(V)\right)^r \times \left(S^1(V)\right)^r \to S^2(V)$$

$$(a_1, \ldots, a_r) \times (b_1, \ldots, b_r) \mapsto \sum_{j=1}^{r} a_j b_j$$

  is a bilinear map from an affine space of dimension $2nr$ to an affine space of dimension $\frac{n(n+1)}{2}$. Now any non-degenerate quadratic form (if $K$ is the field of the complex numbers) is equivalent (after a basis change) to $x_1^2 + \cdots x_n^2$, and the non-degenerate quadratic forms constitute an open subset of $S^2(V)$. So for $\rho_n(f)$ to be contained in the image of the map, the image must contain an open set, and hence $\left(S^1(V)\right) \times \left(S^1(V)\right)^r$ must have higher (or equal) dimension to $S^2(V)$, so that $r \geq \frac{n+1}{4}$.

  Thus $\rho_n(f)$ needs at least $\frac{n+1}{4}$ terms when written as a sum of products of pairs of linear forms. Consequently, $f$ can not be written as a finite sum of products of pairs of linear forms.

- page 9, last line: This presupposes of course that the calculation of S-polynomials in $R'$ of the generators of the ideal can be done algorithmically, as is the case when $K$ is computable and the generators of the ideal are given as computable functions $\mathcal{M} \to K$.

- page 9, two last lines: algorithm for solving
- page 11, Lemma 1.4.5 (i): $\langle \mathrm{in}(F)\rangle_{\leq t} = \mathrm{gr}(J)_{\leq t}$
- page 11, Theorem 1.4.6, last two displays: Lpp is missing.
- page 13, line 2 from top: Proposition 1.3.2
- page 14, the algorithm: We assume that no generator $f_i$ is in $K$.
- page 14, the algorithm, third line from bottom: should read **if** $0 \notin \mathrm{Norm}_{F\setminus\{f\}}$, and similarly on the next line.
- page 16: to "split the coefficients" means to regard some variables, previously regarded as coefficients, once more as variables; that is, to perform the isomorphism

  $$R' \simeq U_n[x_1, \ldots, x_n] \simeq U_{n+1}[x_1, \ldots, x_{n+1}],$$

  where $U_n = R' \cap K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]]$.
- page 18: The minimal monomial generators are given by $x_1x_2x_3 \cdots x_{d-\lambda}x_{\lambda-1}^2$ and $x_1x_2x_3 \cdots x_{\mu-6}x_{\mu-4}^6$, for $\lambda \geq 1$ and $\mu \geq 6$.
- page 19, abstract: has an initial ideal.
- page 20, first display:

  $$K[x_1, \ldots, x_{n'}, x_{n'+1}, \ldots, x_n] \twoheadrightarrow \frac{K[x_1, \ldots, x_{n'}, x_{n'+1}, \ldots, x_n]}{(x_{n'+1}, \ldots, x_n)} \simeq K[x_1, \ldots, x_{n'}].$$

- page 20, 6 lines from bottom: determined by the restricted
- page 23, Lemma 2.3.3, line 4 of proof: for some $g \in J$
- page 23, Corollary 2.3.4: The extension is to $R'$.
- page 24, first line: we demand that the $\alpha_{ij}$'s and $\beta_{ij}$'s are different.
- page 24, line 2 and 3: $f = \rho_3(h_1) \in R'$ and $g = \rho_3(h_2) \in R'$.
- page 25, last line of proof: $\widetilde{\mathrm{gr}(I)} = \mathrm{gr}(I) = I$.
- page 27, Definition 2.5.5: "The necessary..."

- page 28, third display from bottom:

$$\mathcal{T}^d \operatorname{gr}(J) = \mathcal{T}^d \rho_{N(d)} (\operatorname{gr}(J))^e \subset \mathcal{T}^d \operatorname{gr}(\rho_{N(d)} (J))^e$$

- page 30, Lemma 2.5.11, last two lines of proof: $\operatorname{Lpp}(\rho_n(P)) = p$ and $\operatorname{Lpp}(\rho_n(Q)) = q$, hence

$$S(\rho_n(P), \rho_n(Q)) = \frac{m}{p}\rho_n(P) - \frac{m}{q}\rho_n(Q)$$

- page 31, second display, first equation:

$$\operatorname{Lpp}(S(\rho_n(f_i), \rho_n(f_j))) = \operatorname{Lpp}(\rho_n(S[f_i, f_j])) = \operatorname{Lpp}(S(f_i, f_j))$$

- page 31, Lemma 2.5.12:

$$(\operatorname{gr}(I))_g = \langle G \rangle_{R'g}$$

- page 31, (2.10): better expressed $\mathcal{T}^d(\operatorname{gr}(\rho_n(J))) \cap \mathcal{M}^n = \mathcal{T}^d(\mathcal{M}^n B_{\leq d}) \subset \mathcal{M}^n$. (2.11) is an equality of subsets of $\mathcal{M}$, and (2.12) is an equality of subsets of $R'$.

- page 32, Corollary 2.5.14, the proof of $(i) \implies (ii)$ is wrong. A correct proof goes as follows: if $\operatorname{gr}(J)$ is finitely generated, then it is generated by finitely many monomials. Let $D$ be the maximal total degree of these finitely many monomials. Then $\hat{N}(d) = \hat{N}(D)$ whenever $d \geq D$ since $A = A_{\leq D}$.

  The second last line of the proof should have $K[x_1, \ldots, x_N]$.

- page 33, first display: $\operatorname{gr}(J)_g \cap \mathcal{M} = \operatorname{gr}(\rho_n(J)) \cap \mathcal{M}_g$

- page 43, Example 3.3.7: The quotient of $K[x_1, x_2]$ by the zero ideal

- page 44, Lemma 3.5.1, second line from bottom: we must have that $\max_i |q_i f_i| > d$, since otherwise $c(f) = \sum c(q_i)c(f_i)$, where the sum is over all $i$ such that $|q_i f_i| = d$.

- page 44, Lemma 3.5.1, last display formula: the condition $\max_i |a_i f_i| > d$ is unnecessary.

- page 47, the second to last paragraph should read: Since $I$ is generic, the quotient $\frac{K[x_1,\ldots,x_n]}{I}$ has (term-wise) minimal Hilbert series among all quotients of $K[x_1, \ldots, x_n]$ by a homogeneous ideal generated by forms of degree $d_1$ to $d_r$. This useful property was shown by Fröberg in [28], and is to be interpreted in the following way: if we write the Hilbert series of the generic quotient as $\sum_{k=0}^{\infty} v_k t^k$ and the Hilbert series of the other algebra as $\sum_{k=0}^{\infty} w_k t^k$, then $v_k \leq w_k$ for all $k$.

- page 52, second paragraph of subsection 3.6: There is a filtration by total degree on [X], $[X] = \cup_{d \geq 0}[X]_{\leq d}$, where $[X]_{\leq d} = \{ m \in [X] \,|\, |m| \leq d \}$. There is a corresponding non-exhaustive filtration on $C[[X]]$; if we write $C[[X]]_{\leq d} = \{ f \in C[[X]] \,|\, \mathrm{Supp}(f) \subset [[X]]_{\leq d} \}$, then $C[[X]]' = \cup_{d \geq 0} C[[X]]_{\leq d}$.

- page 68, second display: $(1-t)^{-n} \prod_{i=1}^{r} (1 - t^{d_i})$. We must also assume that $n \geq r$.

- page 68, last paragraph before section 4.2: The question if

- page 72, last lne before Lemma 4.3.1: a well known fact

- page 78, (4.3): Note that $\frac{\mathrm{Ann}(\bar{g})}{(f)} = \frac{(\bar{h})}{(f)}$, where $h = \gcd(f, g)$.

- page 78, first line below (4.3): To apply this result

- page 79, Example 4.5.5: provides, Poincaré

- page 79, Theorem 4.5.6 a): The explicit (inclusion/exclusion) formula is

$$\sum_{S \subset \{1,\ldots,r\}} (-1)^{|S|} t^{|\mathrm{lcm}(\{ f_i \,|\, i \in S \})|}$$

- page 81, Proposition 4.6.3, lines 3 to 5: Clearly $|h_n| \leq |\rho_n(g)| \leq |g| < \infty$, hence the coherent sequence of the $h_n$'s has bounded total degree. Now $R' \subset \tilde{R} = \varprojlim K[x_1, \ldots, x_n]$ is the subring consisting of precisely the coherent sequences of bounded degrees, which shows that the sequence of the $h_n$'s define an element $h \in R'$.

- page 82, Corollary 4.6.5: This is a corollary to Proposition 4.6.3 rather than to Lemma 4.6.4, since if $f \not| g$ then by contraposition on Proposition 4.6.3 we get that there exists an $n$ such that $\rho_n(f) \not| \rho_n(g)$. For any $\nu \geq n$ we then have that $\rho_\nu(f) \not| \rho_\nu(g)$.

- page 83, Proposition 4.6.8: Not only is no $p_i$ associate to any $q_j$, but no $p_i$ divides any $q_j$, since $f$ and $g$ are relatively prime and hence have no common divisors.

- page 84, second last line: rôle

- page 87, example 5.3.3: We claim that the homogeneous ideal $I = R'_{\geq 1}$ is non-countably generated. It is enough to show that any homogeneous generating set $G$ of $I$ is uncountable. But such a $G$ must contain a $K$-vector basis of the $K$-vector space $R'_1$, which we claim can have no countable basis. To see this, note that $R'_1$ is the set of all infinite linear combinations $\sum_{i=1}^{\infty} c_i x_i$, with $c_i \in K$. If we denote by $V$ the countably-dimensional subvectorspace of $R'_1$ which is spanned by the $x_i$'s, then $V \subset R'_1$, and $R'_1 \simeq V^*$, the dual vector space of $V$. To see this, let $U = \{\, \mu_i | i \in \mathbb{N}^+ \}$ where $\mu_i(x_j) = \delta_{ij}$. Then $U$ is a linearly independent set in $V^*$, and in fact $V^*$ is given by the set of all $\sum_{i=1}^{\infty} c_i \mu_i$, with $c_i \in K$. Hence $R'_1 \simeq V^*$ as $K$ vector spaces.

  The result now follows from the fact that the dual of a countably dimensional vector space never has a countable basis.

- page 99, second last sentence: Computer calculations indicate that there exists many

- page 100, line 4 from top: initial ideals of for instance the generic

- page 101, second last display: $K[x_1, \ldots, x_n] \simeq \frac{K[x_1,\ldots,x_{n+1}]}{(x_{n+1})} \twoheadleftarrow K[x_1, \ldots, x_n]$.

- page 103, Lemma 6.3.1: If $I$ is an ideal of $R'$, then

- page 104, last display: $\sum_{n=0}^{\infty} \sum_{\mathcal{M}^n \setminus \mathcal{M}^{n-1}} c_m m$ with the convention that $\mathcal{M}^{-1} = \emptyset$.

- page 105, Lemma 6.3.6: $f_\nu$ should be $e_l$. Similarly in the paragraph that follows.

- page 105, Remark 6.3.8: $\sum_{n=1}^{\infty} \left( x_n^n - x_{n-1}^{n-1} \right)$

- page 108, last line: whenever $k \geq N$.

- page 109, first line: whenever $k \geq N$,

- page 109, line 4: It follows from Lemma 6.3.6 that $\sum_{k=n}^{\infty} g_{ik}$ is convergent and defines an element in $R'$.

# 1. GRÖBNER BASES IN $R'$

To appear in *Journal of Symbolic Computation* under the name *Gröbner Bases and Normal Forms in a Subring of the Power Series ring on Countably Infinitely Many Variables*


*Abstract*

If $K$ is a field, let the ring $R'$ consist of finite sums of homogeneous elements in $R = K[[x_1, x_2, x_3, \ldots]]$. Then, $R'$ contains $\mathcal{M}$, the free semi-group on the countable set of variables $\{x_1, x_2, x_3, \ldots\}$. In this paper, we generalize the notion of admissible order from finitely generated sub-monoids of $\mathcal{M}$ to $\mathcal{M}$ itself; assume that $>$ is such an admissible order on $\mathcal{M}$. We show that we can define leading power products, with respect to $>$, of elements in $R'$, and thus the initial ideal $\mathrm{gr}(I)$ of an arbitrary ideal $I \subset R'$. If $I$ is what we call a locally finitely generated ideal, then we show that $\mathrm{gr}(I)$ is also locally finitely generated; this implies that $I$ has a finite truncated Gröbner basis up to any total degree. We give an example of a finitely generated homogeneous ideal which has a non-finitely generated initial ideal with respect to the lexicographic initial order $>_{lex}$ on $\mathcal{M}$.


## *1.1 Introduction*

The author was lead to the study of the Gröbner basis theory of the ring $R'$ when investigating the following problem: what is the initial ideal, in particular, with respect to the lexicographic order, of generic ideals? Recall [30, 27, 55] that a generic ideal in a polynomial ring is an ideal generated by generic forms, where furthermore there is no algebraic relation between the coefficients of the generators. When calculating initial ideals of generic ideals of the same type, but in polynomial rings on successively more variables, one notices that they seem to converge to some monomial ideal in infinitely many variables. It is natural to try to study the initial ideal of the ideal generated by generic forms in infinitely many variables, and try to prove that the sequence of initial ideals indeed converge to this ideal.

In this article, we define the ring $R'$, the natural habitat of generic forms in (countably) infinitely many variables, and prove that we may form initial ideals of, in particular, ideals generated by finitely many generic forms. The fact that this initial ideal may be approximated by the initial ideals of the corresponding ideals in polynomial rings with finitely many variables, is the topic of a forthcoming article [76].

## 1.2 Preliminaries

If $S$ is a ring, and $A \subset S$ is a subset, then $\langle A \rangle_S$ denotes the ideal in $S$ that $A$ generates. Similarly, if $M$ is a monoid, and $A \subset M$ is a subset, then $\langle A \rangle$ denotes the semi-group ideal $\{ ma \mid a \in A, \ m \in M \}$. All rings and monoids under consideration will be commutative.

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

### 1.2.1 Power products

Let $\mathcal{N} = \coprod_{\mathbb{N}^+} \mathbb{N}$. For $\boldsymbol{\alpha} \in \mathcal{N}$, a *power product* (or *monomial*) $\mathbf{x}^{\boldsymbol{\alpha}}$ in the variables $x_1, x_2, \ldots$ is defined by $\mathbf{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^{\infty} x_i^{\alpha_i}$. The set of power products in the variables $x_1, x_2, \ldots$ is a monoid under the obvious multiplication. It is denoted $\mathcal{M} = \{ \mathbf{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{N} \}$.

For $\boldsymbol{\alpha} \in \mathcal{N}$, the *total degree* of $\boldsymbol{\alpha}$ is $|\boldsymbol{\alpha}| = \sum_{i=1}^{\infty} \alpha_i$. For a power product $\mathcal{M} \ni m = \mathbf{x}^{\boldsymbol{\alpha}}$, the total degree is $|m| = |\boldsymbol{\alpha}|$. The *support* of $m$ is defined by $\operatorname{Supp}(m) = \{ i \in \mathbb{N}^+ \mid x_i \mid m \}$. For $m \neq 1$, this set is non-empty, and has a maximum which is denoted $\operatorname{maxsupp}(m)$, the *maximal support* of $m$. We use the convention that $\operatorname{maxsupp}(1) = 0$.

For $n \in \mathbb{N}$, define

$$\mathcal{M}^n = \{ m \in \mathcal{M} \mid \operatorname{maxsupp}(m) \leq n \}$$
$$\mathcal{M}[n] = \{ \mathbf{x}^{\boldsymbol{\alpha}} \mid i \leq n \implies \alpha_i = 0 \}.$$

Note that $\mathcal{M}^0$ is the trivial semi-group, and that $\mathcal{M}[0] = \mathcal{M}$. The monoids $\mathcal{M}^n$ and $\mathcal{M}[n]$ may be regarded as sub-monoids of $\mathcal{M}$. Furthermore, $\mathcal{M}$ is isomorphic to $\mathcal{M}[n]$ via

$$\mathcal{M} \ni \prod_{i=1}^{\infty} x_i^{\alpha_i} \mapsto \prod_{i=1}^{\infty} x_{i+n}^{\alpha_i} \in \mathcal{M}[n].$$

### 1.2.2 The rings $R$ and $R'$

Let $K$ be a field, and denote by $R$ the ring of power series in countably many variables, with coefficients in $K$; $R = K[[x_1, x_2, \ldots, ]]$. For any positive integer $n$, the power series ring $K[[x_1, \ldots, x_n]]$ is both a sub-algebra and a quotient of $R$, since[1] $\frac{R}{B_n} \simeq K[[x_1,\dots,x_n]]$, where $B_n$ is the ideal of $R$ generated by all power series in $K[[x_{n+1},x_{n+2},x_{n+3},\dots]]$ of total degree $\geq 1$ and with zero constant term. We define a $K$-algebra epimorphism $\rho_n$, the $n$*'th truncation homomorphism*, by means of the composite

$$R \twoheadrightarrow \frac{R}{B_n} \simeq K[[x_1,\dots,x_n]].$$

For $n \in \mathbb{N}$, denote by $R_n$ the $K$-vector space $\{\sum_{\substack{\alpha\in\mathcal{N}\\|\alpha|=n}} \mathbf{c}_\alpha \mathbf{x}^\alpha\}$. Note that $R_0 = K$, and that $R = \prod_{n\in\mathbb{N}} R_n$. The ring $R'$ is defined as the smallest $K$-sub-algebra of $R$ that contains all homogeneous elements; $R' = \coprod_{n\in\mathbb{N}} R_n$. Note that for $n \in \mathbb{N}^+$, $\rho_n(R') = K[x_1,\dots,x_n]$. The ring $R'$ is of interest partly because it allows for a generalization of the notion of *generic form*, a generic form in $K[x_1,\dots,x_n]$ (of some total degree $d$) being a homogeneous element $f = \sum_{m\in\mathcal{M}^n,|m|=d} c_m m$ where there are no algebraic (over the prime field of $K$) relations[2]among the coefficients $c_m$. In particular, no coefficients belong to the prime field of $K$, and all $c_m$'s are non-zero. Ideals generated by such elements have been the focus of much study (see for instance [30, 27]). This definition generalizes directly to $R'$, with $f$ expressed as a (not finite!) linear combination of power products in $\mathcal{M}$ with total degree $d$. Note that the infinite polynomial ring $K[x_1,x_2,\dots]$ is not sufficient for this purpose: if $f$ is an element of this ring, then almost every coefficient $c_m$ is zero, which is an element of the prime field. We have that the truncation $\rho_n(f)$ of a generic form in $R'$ is a generic form in $K[x_1,\dots,x_n]$.

Now let $f$ be an arbitrary, non-zero element of $R$,

$$f = \sum_{\alpha\in\mathcal{N}} \mathbf{c}_\alpha \mathbf{x}^\alpha.$$

We define the *set of monomials* of $f$ by

$$\mathrm{Mon}(f) = \{\, \mathbf{x}^\alpha \mid \mathbf{c}_\alpha \neq 0 \,\},$$

and the *total degree* of $f$ by

$$|f| = \sup\{\, |m| \mid m \in \mathrm{Mon}(f) \,\}.$$

For $m = \mathbf{x}^\alpha \in \mathrm{Mon}(f)$ we define the *coefficient of* $m$ *in* $f$ by

$$\mathrm{Coeff}(m,f) = \mathbf{c}_\alpha.$$

[1] We remark that $x_{n+1} + x_{n+2} + x_{n+3} + \cdots \in \langle K[[x_{n+1},x_{n+2},x_{n+3},\dots]] \setminus K\rangle_R$ but not in $(x_{n+1},x_{n+2},x_{n+3},\dots)$, so that $\frac{R}{(x_{n+1},x_{n+2},x_{n+3},\dots)} \not\simeq K[[x_1,\dots,x_n]]$.

[2] Thus, the set $\{\, c_m \mid m \in \mathcal{M}^n, |m| = k \,\}$ is *algebraically independent* ("irreduziert", or "algebraische unabhängig") in the sense of [84]

### 1.2.3 Admissible orders

**Definition 1.2.1.** By an *admissible order* $>$ on $\mathcal{M}$ we mean a total order such that

(A) $m > 1$ for all $m \in \mathcal{M} \setminus \{1\}$.

(B) $p > p' \implies mp > mp'$ for all $m, p, p' \in \mathcal{M}$.

(C) $x_1 > x_2 > x_3 > \cdots$

**Example 1.2.2.** As an example of an admissible order on $\mathcal{M}$, the *lexicographic* order is defined by $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$ iff there exist an $n \in \mathbb{N}^+$ such that $\alpha_n > \beta_n$ and for all $k < n$ we have that $\alpha_k = \beta_k$.

**Lemma 1.2.3.** *If* $n \in \mathbb{N}^+$, $m \in \mathcal{M}^n \setminus \{1\}$ *and* $p \in \mathcal{M}[n]$, *and furthermore* $|m| \geq |p|$, *then* $m > p$ *for any admissible order* $>$ *on* $\mathcal{M}$.

*Proof.* Denote by $V$ the set $\{x_1, \ldots, x_n\}$ and by $W$ the set $\{x_{n+1}, x_{n+2}, \ldots\}$. Clearly, if $v \in V$ and $w \in W$, then $v > w$. By induction, $\prod_{i=1}^r v_i > \prod_{j=1}^s w_j$ if $r \geq s$.

Now, $m = \prod_{i=1}^r v_i$ with $v_i \in V$ and $r = |m|$. Similarly, $m = \prod_{j=1}^s w_j$ with $w_j \in W$, $s = |p| \leq r$. Therefore, $m > p$. $\square$

If $f \in K[x_1, \ldots, x_n] \setminus \{0\}$ then the set $\mathrm{Mon}(f)$ is finite, and we can find its maximal element, which we call the *leading power product* or *leading monomial* of $f$. It turns out that $R'$ has the essential property that leading power products can be defined for any non-zero element. Moreover, it can be shown that it is the largest $K$-sub-algebra of $R$ with this property.

**Theorem 1.2.4.** *For any admissible order* $>$ *on* $\mathcal{M}$, *and any* $f \in R' \setminus \{0\}$ *the set* $\mathrm{Mon}(f)$ *has a maximal element with respect to* $>$.

*Proof.* First, assume that the assertion holds for homogeneous elements; then $f$ is a finite sum of its homogeneous components, $f = \sum_{i=0}^{|f|} f_i$, where each $\mathrm{Mon}(f_i)$ has a maximal element $p_i$. Clearly $\max_{1 \leq i \leq |f|} p_i$ must be maximal also in $\mathrm{Mon}(f)$.

Hence, we may assume that $f$ is homogeneous of degree $d$. Any homogeneous element of degree 1 has a maximal power product; assume inductively that any homogeneous element in $R'$ of degree $< d$ has a maximal power product. Write $f$ in *distributed form* as $f = \sum_{i=1}^\infty x_i g_i$ where $g_i \in R' \cap K[[x_i, x_{i+1}, \ldots]]$. Thus, $x_1 g_1$ contains all terms that are divisible by $x_1$, and so forth. At least one of the $g_i$'s is non-zero; assume, to simplify notations, that $g_1 \neq 0$. Since $|g_1| < d$, there exists a maximal power product $m_1$ of $g_1$, and $x_1 m_1 \in \mathrm{Mon}(f)$. We claim that any power product $\mathrm{Mon}(f) \ni p > x_1 m_1$ must be divisible by a $x_j$ with $j < N$, where $N = \mathrm{maxsupp}(m_1)$. To see this, we assume, towards a contradiction, that there exist a monomial $p \in \mathrm{Mon}(f) \cap \mathcal{M}[N]$ such that $p > x_1 m_1$. Since $|p| = |x_1 m_1|$, we get from Lemma 1.2.3 that $x_1 m_1 > p$, a contradiction.

This shows that the power products of $\mathrm{Mon}(f)$ that precede $x_1 m_1$ are contained in $S = \mathrm{Mon}(\sum_{i=2}^{N} x_i g_i)$. Let us assume that $t \in \mathrm{Mon}(x_j g_j)$, $1 < j \leq N$. It then follows that $t \leq x_j m_j$, where $m_j$ is the maximal power product in $\mathrm{Mon}(g_j)$ (this maximum exists, by the induction hypothesis). Hence, the maximal element of $\{x_2 m_2, \ldots, x_N m_N\}$ is the maximal power product of $S$.

Therefore, the maximal monomial of $\mathrm{Mon}(f)$ is the maximal element of the finite set $\{x_1 m_1\} \cup \{x_2 m_2, \ldots, x_N m_N\}$. $\square$

*Remark 1.2.5.* One can prove the following, stronger statement: suppose that $>$ is a total order on $\mathcal{M}$ which fulfills properties A and B of Definition 1.2.1. Then, every set $\mathrm{Mon}(f)$, when $f \in R'$, has a maximal element w.r.t. $>$ iff every set $\mathrm{Mon}(g)$, where $g \in R'$, $|g| = 1$ has a maximal element w.r.t. $>$.

**Definition 1.2.6.** If $>$ is an admissible order on $\mathcal{M}$, and $f \in R' \setminus \{0\}$, then the *leading power product*, or *leading monomial*, of $f$ is defined by $\mathrm{Lpp}_>(f) = \mathrm{Lpp}(f) = \max_>(\mathrm{Mon}(f))$. The *leading coefficient* of $f$ is defined by $\mathrm{lc}(f) = \mathrm{Coeff}(\mathrm{Lpp}(f), f)$.

**Definition 1.2.7.** For $F \subset R'$, $\mathrm{in}(F) = \{\, \mathrm{Lpp}(f) \,|\, f \in F \setminus \{0\} \,\}$.

**Lemma 1.2.8.** *If $I$ is an ideal (in $R'$), then $\langle \mathrm{in}(I) \rangle$ is a semi-group ideal in $\mathcal{M}$, and $\langle \mathrm{in}(I) \rangle_{R'}$ is a monomial ideal in $R'$. The latter ideal is also denoted by* $\mathrm{gr}(I)$.

## 1.3 Normal form calculations

The calculations of normal forms are an essential and integral part of any Gröbner basis algorithm. To apply these algorithms in the un-orthodox setting of the algebra $R'$, we need first generalize the procedure for finding normal forms. This generalization is also a topic of considerable interest in itself. We will however restrict our attention to a narrow class of these normal forms, which, for the purpose of Gröbner basis algorithms, suffices.

### 1.3.1 Normal form calculations in $R'$

*Remark 1.3.1.* If $t \in \mathcal{M}$, $f \in R'$, $N = \mathrm{maxsupp}(\mathrm{Lpp}(f))$, then $\mathrm{Lpp}(f) \,|t$ iff $\mathrm{Lpp}(f) \,|t'$, where $t'$ denotes the sub-word of $t$ that is obtained by replacing any occurrence of variables $x_i$ not in $\{x_1, \ldots, x_N\}$ with 1. So $t = t't''$, with $t' \in \mathcal{M}^N$, $t'' \in \mathcal{M}[N]$.

Similarly, if $F \subset R'$ is a set such that

$$S = \sup\{\, \mathrm{maxsupp}(\mathrm{Lpp}(f)) \,|\, f \in F \,\}$$

is finite (in particular, if $F$ is finite), and if $m \in \mathcal{M}$, then $m$ is divisible by $\mathrm{Lpp}(f)$ for some $f \in F$ iff $m'$ is, where $m' \in \mathcal{M}^S$ denotes the $x_1, \ldots, x_S$ part of $m$.

It follows from this observation that we, for the purpose of the normal form calculation, may regard $R'$ as a subring of the polynomial ring

$$K[[x_N, x_{N+1}, \ldots]][x_1, \ldots, x_N],$$

since the variables with indices higher than $N$ will "act as coefficients" during the normal form reductions.

From now on, unless otherwise stated, we assume that $>$ is some fixed admissible order on $\mathcal{M}$, with respect to which leading power products et cetera are formed.

**Proposition 1.3.2.** *Let* $F := \{f_1, \ldots, f_r\} \subset R'$ *consist of monic elements. For* $h \in R'$ *there are* $h_1, \ldots, h_r, \tilde{h} \in R'$ *such that*

$$h = \sum_{i=1}^{r} h_i f_i + \tilde{h}, \quad \mathrm{Lpp}(h_i f_i) \leq \mathrm{Lpp}(h) \textit{ and}$$
$$\tilde{h} = 0 \textit{ or } \mathrm{Mon}(\tilde{h}) \cap \langle \mathrm{in}(F) \rangle = \emptyset.$$

*We say that* $\tilde{h}$ *is a "(polynomial) normal form of* $h$ *with respect to* $F$ *and* $>$*".*

*Proof.* Let

$$N \geq \max_{1 \leq i \leq r} \mathrm{maxsupp}(\mathrm{Lpp}(f_i)),$$

that is, $\mathrm{Lpp}(f_i) \in K[x_1, \ldots, x_N]$ for $1 \leq i \leq r$. Consider $F$ as a subset of $K[[x_{N+1}, x_{N+2}, \ldots]][x_1, \ldots, x_n]$ (note that the elements of $F$ are monic there, too). The result then follows from the (well-known) division algorithm for polynomials with coefficients in commutative rings. □

**Definition 1.3.3.** We denote the set of (polynomial) normal forms of $h$ with respect to $F$ by $\mathrm{Norm}_F(h)$. If $0 \in \mathrm{Norm}_F(h)$, then we say that $h$ reduces to zero with respect to $F$.

**Example 1.3.4.** (Due to Ralf Fröberg.) If $h \in R'$, and

$$F := \{f_1, \ldots, f_r\} \subset R'$$

consists of monic elements, then $h$ may have infinitely many polynomial normal forms with respect to $F$. To demonstrate this, we shall study the normal forms of $h = x_1^2 x_2(x_3 + x_4 + x_5 + \ldots)$ with respect to $F = \{x_1^2 - x_2 x_3, x_1 x_2 - x_3^2\}$. Regarding $R'$ as a subset of

$$S_n := K[[x_{n+1}, x_{n+2}, \ldots]][x_1, \ldots, x_n]$$

we have that

$$h = \left( \sum_{k=3}^{n} x_1^2 x_2 x_k \right) + x_1^2 x_2 \sum_{k=n+1}^{\infty} x_k, \tag{1.1}$$

The normal forms of $x_1^2 x_2 \sum_{k=n+1}^{\infty} x_k$ are

$$\{x_2^2 x_3 \sum_{k=n+1}^{\infty} x_k,\ x_1 x_3^2 \sum_{k=n+1}^{\infty} x_k\}.$$

Each of the $n-2$ first terms in (1.1), that is, terms $x_1^2 x_2 x_k$ with $3 \le k \le n$, have normal forms in $\{x_2^2 x_3 x_k, x_1 x_3^2 x_k\}$; the resulting terms are linearly independent. Thus, we get normal forms for $h$ by choosing one element from each of the pairs, and adding them. It follows that $h$ has exactly $2^{n-1}$ different normal forms in $S_n$, which "lift" to different (polynomial) normal forms in $R'$.

**Definition 1.3.5.** A non-empty set $F \subset R'$ of homogeneous elements is said to be *locally finite* if $\{ f \in F | |f| = k \}$ is finite for all $k$.

**Definition 1.3.6.** A proper homogeneous ideal $I$ of $R'$ is said to be *locally finitely generated* if

$$\forall d : \quad \dim_K \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}} < \infty.$$

Here, $\sum$ denotes (not direct) sum of $K$-vector spaces, $I_d = I \cap R_d$, $R'_d = R_d$. Recall that $R_d$ is the set of all homogeneous power series of degree $d$ in $R$.

**Lemma 1.3.7.** *For a proper homogeneous proper ideal $I$ of $R'$, the following are equivalent:*

*(i)* $I$ *is locally finitely generated.*

*(ii)* $I$ *has a locally finite generating set.*

*Proof.* If $I$ has a locally finite set of generators $F$, then $F$ consists of homogeneous elements, and every set

$$F_t = \{ f \in F | |f| = t \}$$

is finite. Fix a positive integer $d$. Then

$$I_d = \left( R'F \right)_d = \sum_{j=1}^{d} F_j R'_{d-j} = K F_d + \sum_{j=1}^{d-1} F_j R'_{d-j}.$$

Therefore, we can use an noetherian isomorphism (of $K$-vector spaces) to conclude that

$$\mathsf{K}F_d \twoheadrightarrow \frac{\mathsf{K}F_d}{\mathsf{K}F_d \cap \sum_{j=1}^{d-1} F_j R'_{d-j}} \simeq \frac{\mathsf{K}F_d + \sum_{j=1}^{d-1} F_j R'_{d-j}}{\sum_{j=1}^{d-1} F_j R'_{d-j}} = \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}.$$

Since $\mathsf{K}F_d$, by the assumptions, is a finite dimensional $K$-vector space, we must have that

$$\dim_K \frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}} < \infty.$$

Conversely, if $I$ is locally finitely generated, we can for each $d$ "lift" a basis of

$$\frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}$$

to a finite set $F_d \subset I_d$. Assume by induction that $I$ is generated up to degree $d-1$ by $F_{\leq d-1} = \cup_{i=1}^{d-1} F_i$. We must show that $I$ can be generated up to degree $d$ by $F_{\leq d-1} \cup F_d$. To this end, note that the set

$$T := \left\{ hf \,\middle|\, h \in R'_j,\ f \in F_{d-j},\ 1 \leq j \leq d-1 \right\}$$

generates the $K$-vector space $\sum_{j=1}^{d-1} R'_j I_{d-j}$. On the other hand,

$$\frac{I_d}{\sum_{j=1}^{d-1} R'_j I_{d-j}}$$

is finite dimensional, and has a finite basis $\bar{\alpha}_1, \ldots, \bar{\alpha}_r$, which we have lifted to

$$F_d = \{\alpha_1, \ldots, \alpha_r\} \subset I_d.$$

It is now an immediate consequence that $\mathsf{K}F_d + T$ generates the $K$-vector space $I_d$. Therefore, every $h \in I_d$ may be written as

$$h = \sum_{i=i}^{q} f_i h_i + \sum_{j=1}^{r} c_i \alpha_i \qquad f_i \in F_{\leq d-1},\ h_i \in R'_{d-|f_i|},\ c_i \in K \tag{1.2}$$

This shows that $F_{\leq d-1} \cup F_d$ generates $I$ up to degree $d$. □

*Remark 1.3.8.* In a polynomial ring $A$, the elements of degree $d$ (of an homogeneous ideal $I$) that are not generated by elements (in $I$) of degrees $< d$ correspond to non-zero elements in $\frac{I_d}{A_1 I_{d-1}}$. We can use this simpler expression, because $A_d = A_1 A_{d-1}$ for all $d$, and hence

$$A_1 I_{d-1} \supset A_2 I_{d-2} = A_1 A_1 I_{d-2} \supset A_3 I_{d-3} = A_2 A_1 I_{d-3} \supset \cdots$$

For any graded ring, this equality holds if the ring is a polynomial ring over the elements of degree 1; in the literature, one often says that such an $A$ is *naturally graded*.

This condition is *not* fulfilled for the ring $R'$! To see that, for instance, $R'_1 R'_1 \subsetneq R'_2$, consider the element $\sum_{i=1}^{\infty} x_i^2$, which is not expressible as a *finite sum* of products of linear elements.

**Lemma 1.3.9.** *Proposition 1.3.2 holds when $F$ is locally finite instead of finite, if all the other prerequisites for the theorem are fulfilled.*

*Proof.* We may assume that $h$ is homogeneous with total degree $t$. Then $h$ can only be reduced by elements of $F$ with total degree $\leq t$, and we need only consider reductions of $h$ with respect to the finite set of such elements. $\square$

## 1.4 Construction of Gröbner bases

Now that we have developed a satisfactory normal form theory for the algebra $R'$, the construction of Gröbner bases might seem trivial; just do what is done in the polynomial case: start with a finite set of generators, keep adding normal forms of the so-called S-polynomials until no critical pairs remain, and the resulting set will be a Gröbner basis.

There are several difficulties that this, basically sound, method has to overcome. First, we will show that the initial ideal $\text{gr}(I)$ of a finitely generated ideal $I$ of $R'$ need not be finitely generated. Hence, by a Gröbner basis for I we must mean a *possibly infinite* set of generators, whose leading monomials generate $\text{gr}(I)$. It is clear that such a set can not be calculated in a finite number of steps.

Secondly, to prove that a set of generators is a Gröbner basis it is customary to show that every element has a unique normal form with respect to it. The normal form theory, developed in the previous part, only deals with normal forms with respect to a finite set, or a locally finite one. Since locally finite sets by definition are homogeneous, the reader might already have guessed how we plan to proceed: we consider only locally finitely generated ideals. Then, starting with a locally finite set of generators, and adding normal forms of S-polynomials, we can arrange things so that we can calculate the Gröbner basis, *up to any given total degree*, in finite time. Since, for an element of degree $t$, it is only necessary to consider the Gröbner basis up to said degree, we have an algorithm for i.e. solving the ideal membership problem.

### 1.4.1 Homogeneous Gröbner bases in $R'$

**Definition 1.4.1.** For $P, Q \in R'$, let the S-polynomial of $P$ and $Q$ be

$$S(P,Q) = \frac{\mathrm{lc}(Q)\,\mathrm{Lpp}(Q)}{\gcd(\mathrm{Lpp}(P),\mathrm{Lpp}(Q))}P - \frac{\mathrm{lc}(P)\,\mathrm{Lpp}(P)}{\gcd(\mathrm{Lpp}(P),\mathrm{Lpp}(Q))}Q \tag{1.3}$$

**Proposition 1.4.2.** *Let $J$ be an homogeneous ideal in $R'$, and let $F \subset J$ be locally finite (in particular, $F$ consists of homogeneous elements).*

*Then the following conditions on $F$ are equivalent:*

*(i)* $\langle \mathrm{in}(F)\rangle_{R'} = \mathrm{gr}(J)$,

*(ii) If* $Q \in J$ *then* $\mathrm{Norm}_F(Q) = \{0\}$,

*(iii) If* $Q \in J$ *then* $0 \in \mathrm{Norm}_F(Q)$.

*If the conditions are fulfilled, then* $\langle F\rangle_{R'} = J$.

*Proof.* It is easy to modify the proofs of [59], proposition 2.5. Note that the authors assume top-reduced normal forms instead of totally reduced normal forms. □

**Definition 1.4.3.** If the conditions of Proposition 1.4.2 are fulfilled, we say that $F$ is a *Gröbner basis* of $J$.

We will need the following results on "partial" or "truncated" Gröbner bases:

**Proposition 1.4.4.** *Let $J$ be an homogeneous ideal in $R'$, and let $F \subset J$ be a finite set consisting of homogeneous elements. Let $t$ be a positive integer.*

*Then the following conditions on $F$ are equivalent:*

*(i)* $\langle \mathrm{in}(F)\rangle_{R'_{\le t}} = \mathrm{gr}(J)_{\le t}$,

*(ii) If* $Q \in J$, $|Q| \le t$ *then* $\mathrm{Norm}_F(Q) = \{0\}$,

*(iii) If* $Q \in J$, $|Q| \le t$ *then* $0 \in \mathrm{Norm}_F(Q)$.

*If the conditions are fulfilled, then* $\langle F\rangle_{R'_{\le t}} = J_{\le t}$.

*Proof.* The polynomial ring case is treated in [11, Theorem 10.39]; the generalization to $R'$ is straightforward. □

**Lemma 1.4.5.** *Let $J$ be a (not necessarily homogeneous) ideal in the polynomial ring $K[x_1, \ldots, x_n]$, and let $F \subset J$ be a finite set consisting of (not necessarily homogeneous) elements. Let $t$ be a positive integer. Suppose that the admissible order $>$ is* degree-compatible*, that is,*

$$|m| > |m'| \implies m > m'.$$

*Then the following assertions are equivalent:*

*(i)* $\langle F\rangle_{K[x_1,\ldots,x_n]_{\leq t}} = \mathrm{gr}(J)_{\leq t}$,

*(ii) If* $P, Q \in J$, $|S(P,Q)| \leq t$ *then* $0 \in \mathrm{Norm}_F(S(P,Q))$*; if* $P, Q \in J$, $|S(P,Q)| > t$ *then either* $0 \in \mathrm{Norm}_F((S(P,Q))$ *or all elements of* $\mathrm{Norm}_F(S(P,Q))$ *have total degree* $> t$.

*If the conditions are fulfilled, then* $\langle F\rangle_{K[x_1,\ldots,x_n]_{\leq t}} = J_{\leq t}$.

The following theorem is the main result of this paper:

**Theorem 1.4.6.** *Let* $I$ *be a homogeneous ideal of* $R'$*, and let* $G$ *be a finite set of monic, homogeneous elements in* $R'$ *that generates* $I$ *up to degree* $t$*. Then, the following assertions are equivalent:*

*(i)* $P, Q \in G$, $|S(P,Q)| \leq t \implies 0 \in \mathrm{Norm}_G(S(P,Q))$,

*(ii)* $\mathrm{gr}(I)_{\leq t} = \langle \mathrm{in}(G)\rangle_{R'_{\leq t}}$.

*It follows that a locally finite set* $F$*, consisting of monic elements, is a Gröbner basis of a locally finitely generated ideal* $J$ *iff every S-polynomial* $S(P,Q)$ *,* $P, Q \in F$ *reduces to zero with respect to* $F$.

*Proof.* (ii) $\implies$ (i): Since $S(P,Q) \in I$, $|S(P,Q)| \leq t$, Proposition 1.4.4 implies that $0 \in \mathrm{Norm}_G(S(P,Q))$.

(i) $\implies$ (ii): Since $I$ and $G$ are homogeneous, $\mathrm{gr}(I)$ and $\mathrm{in}(G)$ are not changed if we replace the admissible order $>$ with the degree-compatible order $>_{tot}$ defined by $m >_{tot} m'$ if $|m| > |m'|$ or if $|m| = |m'|$ and $m > m'$. We can therefore assume that $>$ is degree-compatible.

It is enough (by induction) to prove the inclusion $\mathrm{gr}(I)_t \subset \langle \mathrm{in}(G)\rangle_{R'}$ . Choose a (monic, homogeneous) $h \in I_t \setminus \{0\}$. We must prove that $\mathrm{Lpp}(h) \in \langle \mathrm{in}(G)\rangle_{R'}$ .

Let $N$ be the necessary number of "active variables": that is, $N$ indicates which polynomial ring

$$S_N := K[[x_{N+1}, x_{N+2}, \ldots]][x_1, \ldots, x_N]$$

we will embed $R'$ into. We demand four things from $N$: first,

$$N \geq \max_{Q \in G} \mathrm{maxsupp}(Q),$$

secondly, if $P, Q \in G$ then

$$N \geq \mathrm{maxsupp}(S(P,Q)).$$

The third demand is this: we know that for every pair $P, Q \in G$, if the S-polynomial $S(P,Q)$ has total degree $\leq t$, then it reduces to zero with respect to G. Recalling the proof of Proposition 1.3.2, we get that there is some integer $n$, depending on $P$ and $Q$, such that the normal form $0$ was formed in the polynomial ring $S_n$. We demand that $N$ is greater than all of these $n$'s, for some choice of normal form reductions to zero of $S(P,Q)$, for every pair $P,Q \in G$ such that $|S(P,Q)| \leq t$.

Since G consists of homogeneous elements, the normal form, with respect to $G$, of an S-polynomial $S(P,Q)$, $P,Q \in G$, $|S(P,Q)| > t$, is either zero or has total degree $> t$. We demand (the fourth demand) that this is also the case when we "embed" everything into the polynomial ring $S_N$. If $N$ is too small, then we could have that in the leading power product of the normal form, some variables occuring were regarded as coefficients, which could lower the total degree of the normal form so that it became $\leq t$, resulting in a new minimal monomial generator for the initial ideal of degree $\leq t$. By considering the reductions to normal forms of the finitely many $S(P,Q)$, $P,Q \in G$, $|S(P,Q)| > t$, and choosing sufficiently many "active variables" so that when the reduction chain is regarded as a reduction chain in $S_N$, the normal form of $S(P,Q)$ (in $S_N$) always has the same total degree as $S(P,Q)$ (for some choice of a normal form for each S-polynomial), we avoid this calamity.

Injecting $S_N$ into

$$T_N := K((x_{N+1}, x_{N+2}, \dots))[x_1, \dots, x_N],$$

where the field $K((x_{N+1}, x_{N+2}, \dots))$ is the field of fractions of the domain $K[[x_{N+1}, x_{N+2}, \dots]]$, we are sure that we can apply standard Gröbner basis techniques. Note that the elements of $G$ are monic even as elements of $T_N$, so we need never divide with a variable $x_j$ when performing normal form calculations; thus the computations actually take place within $S_N$. Neither $h$, the element of $I_t \setminus \{0\}$ chosen above, nor the elements of $G$ need be homogeneous, when regarded as elements of $T_N$ (since some variables get demoted to coefficients when passing from $R'$ to $S_N$, and therefore homogeneous elements of $R'$ may become non-homogeneous when regarded as elements of $S_N$), but that is a small matter: the important thing is that the leading power products are preserved. Furthermore, inside $T_N$, all S-polynomials of degree $\leq t$ reduce to $0$ with respect to G. We also have that all S-polynomials of degree $> t$ either reduce to zero or have normal forms with total degree $> t$.

Because of this, the image of G in $T_N$ is a partial Gröbner basis, up to degree $t$, of the extension of the ideal I to the ideal $I^e \subset T_n$, by Lemma 1.4.5. It is now clear that when $h$ is regarded as an element of $S_N$, then $\mathrm{Lpp}(h) \in \langle \mathrm{in}(G) \rangle_{S_N}$. Since $N$ is taken large enough, this implies that when we once more regard $h$ as an element of $R'$, then $\mathrm{Lpp}(h) \in \langle \mathrm{in}(G) \rangle_{R'}$.

The general result follows easily from the result on "partial" Gröbner bases. $\square$

### 1.4.2 A Gröbner basis algorithm in $R'$

The most natural way, perhaps, to extend the usual Gröbner basis algorithm in polynomial rings, is to use the normal form algorithm sketched in 1.3.2, and try to work directly in $R'$. That is, we start with a locally finite generating set of our locally finitely generated ideal $I$, and then proceed, degree by degree, to add normal forms of S-polynomials of the generators; here, the normal forms are elements in $R'$.

We can also work within the polynomial rings

$$K((x_{n+1}, x_{n+2}, \dots))[x_1, \dots, x_n],$$

successively promoting "constants" to "variables" as the need arises. The resulting algorithm would not differ from the one we describe; it is merely another way of viewing the original one. In Section 1.5.1 we sometimes take this view when we talk about "splitting the coefficients" and "active variables".

In either case, the algorithm works with homogeneous in-data, and uses a variant of the so called *normal selection strategy* as defined in [20] and [33]; it uses this strategy, but the admissible order $>_{tot}$ defined by $m >_{tot} p \iff |m| > |p| \vee (|m| = |p| \wedge m > p)$ is used for comparisons. Note that every element in the (preliminary) Gröbner basis will be homogeneous, and hence that every comparison of monomials will in fact compare monomials of the same total degree, for which $>$ and $>_{tot}$ coincide. So, the run of the Gröbner basis algorithm, and hence the result, is not changed if we replace $>$ with $>_{tot}$ throughout.

We recall that the normal selection strategy chooses the critical pair $(P, Q)$ with the least lcm(Lpp$(P)$, Lpp$(Q)$). In particular it adds the S-polynomial with lowest total degree first. This is essential, since it guarantees that after each step of the algorithm, the partial Gröbner basis is a locally finite set, and that we, for any total degree $t$, can compute all elements of the Gröbner basis with total degree $\leq t$ in a "finite number of steps" (thus yielding a solution to the ideal membership problem); however, each "step" involves a complicated normal form calculation. In fact, even the seemingly innocuous operation of forming S-polynomials involves infinite operations. Hence, we are not assured that it can be computed in finite time (with for instance a Turing machine). Furthermore, we have not placed any restrictions on the field $K$; it may not be "effectively computable", a technical condition *not* fulfilled for such commonplace rings as $\mathbb{R}$ and $\mathbb{C}$. More on this matter may be found in [74].

To continue with the description of the "algorithm": we add normal forms of S-polynomials as generators, and the normal form sets with respect to the partial Gröbner basis need not be singletons. Therefore, we need to make another choice: what normal form to add. We will tacitly assume the existence of some suitable choice function to facilitate this.

A final remark: the so called Buchberger Criteria can, appropriately modified, be used also in this “algorithm” to avoid unnecessary reductions of S-polynomials.

*Remark 1.4.7.* If $C = (P, Q) \in G$, is a critical pair of elements in $F$, then if the Gröbner basis elements $P$ and $Q$ are changed (as a result of an reduction with respect to a new Gröbner basis element) then the corresponding constituent of $C$ is implicitly assumed to change accordingly. Thus, in a practical implementation, one would save the *pair of indices* of the Gröbner basis elements, rather than the elements themselves.

**Specification:** $F := \mathrm{GBAS}(\{f_1, f_2, f_3, \ldots\})$
Construction of standard basis $F$ of $\langle\{f_1, f_2, f_3, \ldots\}\rangle_{R'}$
**Given:** A locally finite generating set $\{f_1, f_2, f_3, \ldots\} \subset R'$,
homogeneous with $\mathrm{Lpp}(f_i) = m_i$.
**Find:** $F = \cup_{g=1}^{\infty} F_g$, a locally finite set
which is a Gröbner basis for $\langle\{f_1, \ldots, f_r\}\rangle_{R'}$.
**Variables**:
$F_i$ = The Gröbner basis elements of total degree $i$
$G_i$ = Critical pairs which have S-polynomial of total degree $i$.
$F = \cup_{i>0} F_i$ at all times
$G = \cup_{i>0} G_i$ at all times

**for** $g := 1 \ldots \infty$
    **while** $G_g \neq \emptyset$
        Choose a pair $(P, Q) \in G_g$
        $G_g := G_g \setminus (P, Q)$
        **if** $0 \notin \mathrm{Norm}_F(S(P, Q))$
            Choose $h \in \mathrm{Norm}_F(S(P, Q)) \subset R'$
            $h := \frac{h}{\mathrm{lc}(h)}$
            reduce $F_g$ with respect to $h$
            $F_g := F_g \cup \{h\}$
            **forall** $W \in F \setminus \{h\}$
                $d := |\mathrm{lcm}(\mathrm{Lpp}(W), \mathrm{Lpp}(h))|$
                $G_d := G_d \cup \{(W, h)\}$
            **end for**
        **end if**
    **end while**
    **forall** $f \in \{\, f_i \,|\, |f_i| = g \,\}$
        **if** $0 \notin \mathrm{Norm}_F(h)$
            Choose $h \in \mathrm{Norm}_F(f)$
            $h := \frac{h}{\mathrm{lc}(h)}$

Reduce $F_g$ with respect to $h$
$F_g := F_g \cup \{h\}$
**forall** $W \in F \setminus \{h\}$
$d := |\,\mathrm{lcm}(\mathrm{Lpp}(W), \mathrm{Lpp}(h))|$
$G_d := G_d \cup \{(W, h)\}$
**end for**
**end if**
**end for**
**end for**

It is an easy consequence of the previous results, that the output of the "exterior" algorithm is indeed a Gröbner basis:

**Theorem 1.4.8.** *Let* $I$ *be a homogeneous ideal in* $R'$*, generated by a locally finite set* $\{f_1, f_2, f_3, \ldots\}$ *(thus,* $I$ *is locally finitely generated). If* $F = \cup_{g=1}^{\infty} F_g$ *is the output of the "exterior" algorithm, then* $F$ *is a Gröbner basis of* $I$*. Since* $F$ *is a locally finite set, so is the set* $\{\,\mathrm{Lpp}(f) | f \in F\}$*, which generates* $\mathrm{gr}(I)$*. Therefore,* $\mathrm{gr}(I)$ *is locally finitely generated.*

*Remark 1.4.9.* One can easily prove that $F$ has most of the usual properties of a Gröbner basis in a polynomial ring (see [11] and [21]) so that, for instance, normal forms with respect to $F$ are unique. However, it is impossible to decompose the $K$-vector space $R'$ as

$$R' = I \oplus \mathrm{Span}\big(\mathcal{M} \setminus \mathrm{gr}(I)\big).$$

This follows from the fact that

$$\mathrm{Span}\big(\mathcal{M}\big) = K[x_1, x_2, x_3, \ldots] \subsetneq R'.$$

## 1.5 Examples of lexicographic initial ideals of generic ideals

### 1.5.1 A finitely generated initial ideal: two generic quadratic forms

In this section, we will calculate the initial ideal (with respect to the lexicographic order) of the generic ideal spanned by two generic quadratic forms. By "generic ideal", we mean, as in [30, 27], that not only are the generators generic, but they are independent in the sense that the union of their sets of coefficients is algebraically independent. Let therefore $I = (f_1, f_2)$ where $f_1, f_2 \in R_2$ have generic coefficients. There should be no algebraic relation among the non-zero coefficients, nor should these belong to the prime field of $K$. To avoid complicating matters, we will in fact assume that $K = \mathbb{C}$ with prime field $\mathbb{Q}$.

To facilitate computations, we perform a "Gaussian-elimination" step and write the generators as

$$f_1 = {x_1}^2 + a_{1,3}x_1 + \alpha_{2,2}{x_2}^2 + a_{2,3}x_2 + a_3$$
$$f_2 = x_1x_2 + b_{1,3}x_1 + \beta_{2,2}{x_2}^2 + b_{2,3}x_2 + b_3$$

where $a_{1,3} = \sum_{j=3}^{\infty} \alpha_{1,j}x_j$, $a_{2,3} = \sum_{j=3}^{\infty} \alpha_{2,j}x_j$, $a_3 = \sum_{3\le i\le j} \alpha_{i,j}x_ix_j$, $b_{1,3} = \sum_{j=3}^{\infty} \beta_{1,j}x_j$, $b_{2,3} = \sum_{j=3}^{\infty} \beta_{2,j}x_j$ and $b_3 = \sum_{3\le i\le j} \beta_{i,j}x_ix_j$. Following the algorithm, we regard the $f_i$ as elements in $K[[x_3, x_4, \ldots]][x_1, x_2]$ and form the S-polynomial:

$$\begin{aligned} S_{1,2} &= x_2f_1 - x_1f_2 \\ &= -b_{1,3}{x_1}^2 - \beta_{2,2}x_1{x_2}^2 + (a_{1,3} - b_{2,3})\, x_1x_2 - b_3x_1 \\ &\quad + \alpha_{2,2}{x_2}^3 + a_{2,3}{x_2}^2 + a_3x_2. \end{aligned}$$

When we reduce this to normal form, the leading monomial is $(-\beta_{2,2}{\beta_{1,3}}^2 + \beta_{1,3}\beta_{2,3} - \beta_{3,3})x_1x_3^2$. Thus, for the next step of the algorithm we need to add $x_3$ as an active variable. In $K[[x_4, \ldots]][x_1, x_2, x_3]$ the generators can be written as

$$\begin{aligned} f_1 &= {x_1}^2 + \alpha_{1,3}x_1x_3 + a_{1,4}x_1 + \alpha_{2,2}{x_2}^2 + \alpha_{2,3}x_2x_3 \\ &\quad + a_{2,4}x_2 + \alpha_{3,3}{x_3}^2 + a_{3,4}x_3 + a_4 \\ f_2 &= x_1x_2 + \beta_{1,3}x_1x_3 + b_{1,4}x_1 + \beta_{2,2}{x_2}^2 + \beta_{2,3}x_2x_3 \\ &\quad + b_{2,4}x_2 + \beta_{3,3}{x_3}^2 + b_{3,4}x_3 + b_4 \\ f_3 &= x_1{x_3}^2 + Q \end{aligned}$$

where $a_{1,4} = \sum_{j=4}^{\infty} \alpha_{1,j}x_j$, $a_{2,4} = \sum_{j=4}^{\infty} \alpha_{2,j}x_j$, $a_{3,4} = \sum_{j=4}^{\infty} \alpha_{3,j}x_j$, $a_4 = \sum_{4\le i\le j} \alpha_{i,j}x_ix_j$, $b_{1,4} = \sum_{j=4}^{\infty} \beta_{1,j}x_j$, $b_{2,4} = \sum_{j=4}^{\infty} \beta_{2,j}x_j$, $b_{3,4} = \sum_{j=4}^{\infty} \beta_{3,j}x_j$, $b_4 = \sum_{4\le i\le j} \beta_{i,j}x_ix_j$ and $Q$ is a rather longish expression that is omitted in the interest of brevity. Now we form the S-polynomial of $f_1$ and $f_3$ in $K[[x_4, \ldots]][x_1, x_2, x_3]$ and reduce it with respect to $\{f_1, f_2, f_3\}$. The resulting expression is somewhat long, so we give here only the leading term, which is

$$-\frac{\beta_{2,2}\left({\beta_{2,2}}^2 + \alpha_{2,2}\right)}{\beta_{2,2}{\beta_{1,3}}^2 - \beta_{1,3}\beta_{2,3} + \beta_{3,3}} x_2^4.$$

Since the leading coefficient lies in $K$, we need not split the coefficients. We add $f_4$, a monic polynomial in $K[[x_4, \ldots]][x_1, x_2, x_3]$ with leading monomial $x_2^4$, to our basis. Forming $S(f_2, f_3)$, we find that it reduces to $0$ with respect to $\{f_1, f_2, f_3, f_4\}$. We are now done, since $S(f_i, f_4)$ must , for $i = 1, 3$, reduce to $0$ with respect to $\{f_1, f_2, f_3, f_4\}$ by *Buchbergers first criterion*, and $S(f_2, f_4)$ reduce to $0$ as well. Lifting the result back to $R'$, we have that $\mathrm{gr}(I) = (x_1^2, x_1x_2, x_1x_3^2, x_2^4)$.

### 1.5.2 A finitely generated ideal having non-finitely generated initial ideal: the generic ideal generated by a quadratic and a cubic form

If we modify the previous example, studying the generic ideal $I = (f, g)$ where $f$ is a quadratic generic form and $g$ is a cubic generic form, then, the (lexicographic) initial ideal $\mathrm{gr}(I)$ is locally finitely generated but not finitely generated[3]. In fact, the initial ideal $\mathrm{gr}(I)$ is generated by

$$x_1^2,\ x_1x_2^2,\ x_1x_2x_3^2,\ x_1x_2x_3x_4^2,\ x_1x_2x_3x_4x_5^2,\ x_2^6,\\ x_1x_2x_3x_4x_5^2,\ x_1x_3^6,\ x_1x_2x_3x_4x_5x_6^2,\ x_1x_2x_4^6,\ x_1x_2x_3x_4x_5x_6x_7^2,\ldots$$

where, for a total degree $t \geq 9$, the minimal monomial generators of degree $t$ are

$$x_1x_2\cdots x_{t-6}x_{t-4}^6, \quad x_1x_2\cdots x_{t-1}x_t^2.$$

This initial ideal provides some information on the initial ideals of the restricted ideals $\rho_n(I) \subset K[x_1,\ldots,x_n]$ of $I$: these are ordinary generic ideals generated by a quadratic and a cubic form. Their initial ideals have been studied by Alyson Reeves [60]. We tabulate the first of these initial ideals in Table 1.1.

The author has proved [76], that, for all locally finitely generated ideals $J$, the relation $\lim_{n\to\infty} \mathrm{gr}(\rho_n(J)) = \mathrm{gr}(J)$ holds, in the following sense:

$$\forall d : \exists N(d) : n > N(d) \implies \mathrm{gr}(J)_{\leq d} = \Big(\big\langle \mathrm{gr}\big(\rho_n(J)\big)\big\rangle_{R'}\Big)_{\leq d}.$$

So the initial ideals of all restricted ideals determine $\mathrm{gr}(J)$; the converse, on the other hand, does not hold in general: studying Table 1.1, we see that $\mathrm{gr}(\rho_2(I))$ has the minimal monomial generator $x_2^4$; this "tail", which may be regarded as an effect of the truncation to two variables (the corresponding generator of the same degree for $\mathrm{gr}(I)$ is $x_1x_2x_3^2$) is impossible to detect from the study of $\mathrm{gr}(I)$ alone.

## 1.6 Acknowledgements

I would like to thank Jörgen Backelin and Ralf Fröberg (and the referees!) for their patience in scrutinizing the versions of this paper that preceded the present one, and for their helpful suggestions.

[3] We have not proved this, but computer calculations makes it probable that it is so.

| Degree | $gr(\rho_2(I))$ | $gr(\rho_3(I))$ | $gr(\rho_4(I))$ |
|---|---|---|---|
| 2 | $x_1^2$ | $x_1^2$ | $x_1^2$ |
| 3 | $x_1x_2^2$ | $x_1x_2^2$ | $x_1x_2^2$ |
| 4 | $x_2^4$ | $x_1x_2x_3^2$ | $x_1x_2x_3^2$ |
| 5 | | $x_1x_3^4$ | $x_1x_2x_3x_4^2$ |
| 6 | | | $x_1x_2x_4^4$ |
| 6 | | $x_2^6$ | $x_2^6$ |
| 7 | | | |
| 7 | | | $x_1x_3^6$ |
| 8 | | | |
| 8 | | | |
| 9 | | | |
| 9 | | | |
| 10 | | | |
| 10 | | | |

***Tab. 1.1:*** *Initial ideals of restricted ideals of the generic ideal generated by a quadratic and a cubic form, 2 to 4 variables*

| Degree | $gr(\rho_5(I))$ | $gr(\rho_6(I))$ | $gr(\rho_7(I))$ |
|---|---|---|---|
| 2 | $x_1^2$ | $x_1^2$ | $x_1^2$ |
| 3 | $x_1x_2^2$ | $x_1x_2^2$ | $x_1x_2^2$ |
| 4 | $x_1x_2x_3^2$ | $x_1x_2x_3^2$ | $x_1x_2x_3^2$ |
| 5 | $x_1x_2x_3x_4^2$ | $x_1x_2x_3x_4^2$ | $x_1x_2x_3x_4^2$ |
| 6 | $x_1x_2x_3x_4x_5^2$ | $x_1x_2x_3x_4x_5^2$ | $x_1x_2x_3x_4x_5^2$ |
| 6 | $x_2^6$ | $x_2^6$ | $x_2^6$ |
| 7 | $x_1x_2x_3x_5^4$ | $x_1x_2x_3x_4x_5x_6^2$ | $x_1x_2x_3x_4x_5x_6^2$ |
| 7 | $x_1x_3^6$ | $x_1x_3^6$ | $x_1x_3^6$ |
| 8 | | $x_1x_2x_3x_4x_6^4$ | $x_1x_2x_3x_4x_5x_6x_7^2$ |
| 8 | $x_1x_2x_4^6$ | $x_1x_2x_4^6$ | $x_1x_2x_4^6$ |
| 9 | | | $x_1x_2x_3x_4x_5x_7^4$ |
| 9 | | $x_1x_2x_3x_5^6$ | $x_1x_2x_3x_5^6$ |
| 10 | | | |
| 10 | | | $x_1x_2x_3x_4x_6^6$ |

***Tab. 1.2:*** *Initial ideals of restricted ideals of the generic ideal generated by a quadratic and a cubic form, 5 to 7 variables*

# 2. INITIAL IDEALS OF TRUNCATED HOMOGENEOUS IDEALS

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*Abstract*

Denote by $R$ the power series ring in countably many variables over a field $K$; then $R^{'}$ is the smallest sub-algebra of $R$ that contains all homogeneous elements. It is a fact that a homogeneous, finitely generated ideal $J$ in $R^{'}$ have an initial ideal $\mathrm{gr}(J)$, with respect to an arbitrary admissible order, that is *locally finitely generated* in the sense that

$$\dim_K \left( \frac{\mathrm{gr}(J)_d}{\sum_{j=1}^{d-1} R_j^{'} \, \mathrm{gr}(J)_{d-j}} \right) < \infty$$

for all total degrees $d$. Furthermore, $\mathrm{gr}(J)$ is locally finitely generated even under the weaker hypothesis that $J$ is homogeneous and locally finitely generated.

In this paper, we investigate the relation between $\mathrm{gr}(J)$ and the sequence of initial ideals of the "truncated" ideals

$$\rho_n(J) \subset K[x_1, \ldots, x_n].$$

It is shown that $\mathrm{gr}(J)$ is reconstructible from said sequence. More precisely, it is shown that for all $g$ there exists an $N(g)$ such that

$$\mathcal{T}^g \, \mathrm{gr}(J) = \mathcal{T}^g \, \mathrm{gr}(\rho_n(J))^e$$

whenever $n > N(g)$; here $\mathcal{T}$ denotes the total-degree filtration.

## 2.1 Introduction

The starting point for the investigations that lead to this article was the question: "what is the relation between the initial ideal of an ideal generated by $m$ forms in $n$ variables, and the initial ideal of the *truncation* of the ideal to the polynomial ring in $n^{'}$ variables?". Recall that a *form* is a homogeneous polynomial. By the truncation of a polynomial in $n$ variables to one in $n^{'}$ variables we mean the polynomial that is obtained by removing any monomial divisible by a variable with index greater than $n'$. This is of course the same as taking the image under the quotient epimorphism

$$\frac{K[x_1, \ldots, x_{n'}, x_{n'+1}, \ldots, x_n]}{(x_{n'+1}, \ldots, x_n)} \simeq K[x_1, \ldots, x_{n'}].$$

Computing a large number of examples, in different monomial orderings, one notices that the following seems to hold: the initial ideals of the ideals above will differ in high degrees, but coincide in low degrees.

Conversely, if we fix a degree $d$, and assume that $n$ is very, very large, then, varying $n'$, we note that for sufficiently large $n'$ we have that the initial ideals of the ideals coincide up to degree $d$. So, the initial ideals of these restricted ideals is made up of two parts: the "variable-independent" components of lower degree, and the "tail", which varies with $n'$.

Now, let us assume that $n = \infty$, that is, the original ideal $J$ is generated by "generalized forms" with infinitely (countably) many variables.

In [75] the theory for calculating initial ideals for $J$ *inside* the pertinent ring (called $R'$) is developed. It is natural to ask whether this ideal can be approximated degree-wise in the fashion outlined above: that is, if we for a fixed degree $d$ can find an $N(d)$ such that, for any $n \geq N(d)$, the minimal monomial generators of the initial ideal $\text{gr}(J)$ of $J$ and the minimal monomial generators of $\text{gr}(\rho_n(J))$, the initial ideal of the truncation of $J$, coincide up to degree $d$.

This article answers this question affirmatively. In fact, it is showed that we may take $J$ to be any homogeneous locally finitely generated ideal, by which we mean that

$$\dim_K \frac{J_d}{\sum_{j=1}^{d-1} R'_j J_{d-j}} < \infty$$

for all $d$, and conclude that there exists an $\hat{N}(d)$ such that

$$\text{gr}(J)_d = \text{gr}(\rho_n(J))^e_d$$

whenever $n > \hat{N}(d)$. An immediate consequence of this result (which is stated in Theorem 2.5.13) is that the initial ideal of $J$ is completely determined by the the restricted ideals of $J$.

## 2.2 Preliminaries

All rings and semi-groups under consideration will be commutative. Let $K$ be a field, and let $R = K[[x_1, x_2, x_3, \ldots]]$ be the power series ring over $K$ on a denumerable family of variables. Define $R'$ to be the the smallest sub-ring of $R$ that contains all homogeneous (with respect to total degree) elements. Let $\mathcal{M}$ be the free (commutative) monoid on the variables $x_1, x_2, x_3, \dots$. Regarding an element $m \in \mathcal{M}$ as a finitely supported map $\mathbb{N}^+ \to \mathbb{N}$, we define $\mathrm{Supp}(m) \subset \mathbb{N}^+$, and put

$$\mathrm{maxsupp}(m) = \max \mathrm{Supp}(m).$$

Then, for each $n$, we can define the subsemigroups

$$\mathcal{M}^n := \{\, m \in \mathcal{M} | \mathrm{maxsupp}(m) \leq n \,\}.$$

If $>$ is an *admissible order* on $\mathcal{M}$, that is, a total order that respects the multiplicative structure (so that 1 is the smallest element, and $m > m' \implies tm > tm'$) and is such that $x_1 > x_2 > x_3 > \cdots$, it is shown in [75] that for each $f \in R'$, the set $\mathrm{Mon}(f) \subset \mathcal{M}$ of all monomials (also called power products) of $f$ have a maximal element with respect to $>$. This monomial is called the *leading monomial* of $f$ and is denoted by $\mathrm{Lpp}(f)$.

Let $I$ be an ideal of $R'$. The *initial ideal* $\mathrm{gr}(I)$ is the monomial ideal generated by all leading monomials of elements in $I$.

We denote by $|f|$ the total degree of $f$. There is a natural filtration on $R$ by

$$\mathcal{T}^k R = \{\, f \in R | |f| \leq k \,\}.$$

This restricts to a filtration on $R'$, as well as on $I$, and $\mathrm{gr}(I)$. In fact, $R'$ is isomorphic to the graded associated ring of $R$ w.r.t this filtration, so $R'$ is positively graded; $R' = \coprod_{i \geq 0} R_i$, whereas $R = \prod_{i \geq 0} R_i$. We denote by $I_d = R_d \cap I$ the set of homogeneous elements of total degree $d$ in $I$.

For any positive integer $n$, the power series ring $K[[x_1, \dots, x_n]]$ is both a subalgebra and a quotient of $R$, since

$$R/B_n \simeq K[[x_1, \dots, x_n]],$$

where $B_n \subset R$ is the ideal generated by

$$K[[x_{n+1}, x_{n+2}, x_{n+3}, \dots]]_{\geq 1} = \\ = \{\, f \in R | f(0, \dots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \dots) = 0 \,\}.$$

Therefore, we can define an $K$-algebra epimorphism $\rho_n$ (the $n$'th *truncation homomorphism*) by the composite

$$R \twoheadrightarrow R/B_n \simeq K[[x_1, \dots, x_n]].$$

Note that for $m \in \mathcal{M}$, $\rho_n(m) = m$ if $\mathrm{maxsupp}(m) \leq n$, and $0$ otherwise. Thus $\rho_n(\mathcal{M}) = \mathcal{M}^n \cup \{0\}$.

Clearly, the inverse limit of the inverse system

$$K \xleftarrow{\rho_0} K[[x_1]] \xleftarrow{\rho_1} K[[x_1, x_2]] \xleftarrow{\rho_2} K[[x_1, x_2, x_3]] \xleftarrow{\rho_3} \cdots \tag{2.1}$$

is equal to $R$. If we consider only the coherent sequences of bounded degree, we find that these elements form a ring isomorphic to $R'$. On the other hand, since for each $n$, we have that

$$\rho_n\left(R'\right) = K[x_1, \ldots, x_n],$$

we also have that

$$\rho_n\left(K[x_1, \ldots, x_{n+1}]\right) = K[x_1, \ldots, x_n],$$

so the inverse system (2.1) contains as a subsystem all polynomial rings $K[x_1, \ldots, x_n]$. We put $\tilde{R} := \varprojlim K[x_1, \ldots, x_n]$; it is easy to show that

$$\tilde{R} \cong \{\, f \in R \,|\, \rho_n(f) \in K[x_1, \ldots, x_n] \text{ for all } n \in \mathbb{N} \,\}.$$

For completeness, we consider also the direct limits (under inclusion) of the polynomial rings $K[x_1, \ldots, x_n]$, and the direct limit of of the power series rings $K[[x_1, \ldots, x_n]]$. It is not hard to prove that

$$\varinjlim K[x_1, \ldots, x_n] \subsetneq R' \subsetneq \tilde{R} \subsetneq R$$

whereas $\varinjlim K[[x_1, \ldots, x_n]]$ contains $\varinjlim K[x_1, \ldots, x_n]$ strictly, but does not contain, nor is it contained in, the ring $R'$.

If $I \subset R'$ is an ideal, then so is $\rho_n(I) \subset K[x_1, \ldots, x_n]$ for any positive $n$. The latter ideal is said to be the $n$-th *truncation* of $I$. We also say that it is a *truncation* of $I$; furthermore, we call the inverse, surjective system

$$\rho_1(I) \xleftarrow{\rho_1} \rho_2(I) \xleftarrow{\rho_2} \rho_3(I) \xleftarrow{\rho_3} \rho_4(I) \xleftarrow{\rho_4} \cdots \tag{2.2}$$

a *co-filtration* of $I$; we use the same term for the inverse (not surjective!) system that we get by extending (2.2) to $R'$ by means of the natural injections $K[x_1, \ldots, x_n] \hookrightarrow R'$.

## 2.3 Truncation and initial ideals

Of critical importance when approximating $\mathrm{gr}(I)$ with the monomial ideals $\mathrm{gr}(\rho_n(I))$ will be the relation between these and the ideals $\rho_n(\mathrm{gr}(I))$. In this section, we show that although the operations of truncation and forming initial ideals does not commute, there is a useful relation between the two that we can exploit; this relation is quite similar to the way that "specialization", or more generally, extension of scalars, interacts with the operation of forming initial ideals (see [35, 7]).

### 2.3.1 Truncations and leading monomials

**Lemma 2.3.1.** *If* $f \in R' \setminus \{0\}$ *and* $n = \operatorname{maxsupp}\operatorname{Lpp}(f)$ *then we have that* $\operatorname{Lpp}(f) = \operatorname{Lpp}(\rho_k(f))$ *whenever* $k \geq n$.

*Proof.* For all $k$, we have that

$$\operatorname{Mon}(\rho_k(f)) \subset \operatorname{Mon}(f)$$

and hence that $\operatorname{Lpp}(\rho_k(f)) \leq \operatorname{Lpp}(f)$. If $k \geq n$ then

$$\operatorname{Lpp}(f) \in \operatorname{Mon}(\rho_k(f)),$$

hence $\operatorname{Lpp}(\rho_k(f)) = \operatorname{Lpp}(f)$. □

**Lemma 2.3.2.** *If* $f \in R'$, $m = \operatorname{Lpp}(f)$, $p \in \mathcal{M}$ *then* $\operatorname{Lpp}(fp) = mp$.

**Lemma 2.3.3.** *If* $J$ *is an ideal in* $R'$*, then* $\rho_n(\operatorname{gr}(J)) \subset \operatorname{gr}(\rho_n(J))$ *for all* $n$*. If* $J$ *is a monomial ideal then equality holds.*

*Proof.* Since we are comparing monomial ideals, we need only check the inclusion for monomials. Let $m$ be a typical element of $\rho_n(\operatorname{gr}(J)) \cap \mathcal{M}$, that is, $m \in \mathcal{M}^n$, $m = \operatorname{Lpp}(f)$ and $f \in J$. We must prove that $m \in \operatorname{gr}(\rho_n(J))$, that is, that $\operatorname{Lpp}(f) = \operatorname{Lpp}(\rho_n(g))$ for some $g \in R'$. By Lemma 2.3.1, $g = f$ suffices.

When $J$ is a monomial ideal, so is $\rho_n(J)$, hence

$$\rho_n(\operatorname{gr}(J)) = \rho_n(J) = \operatorname{gr}(\rho_n(J)).$$

□

The following corollary is immediate:

**Corollary 2.3.4.** *If* $J$ *is an ideal in* $R'$*, then*

$$\rho_n(\operatorname{gr}(J))^e \subset \operatorname{gr}(\rho_n(J))^e$$

*for all* $n$*. If* $J$ *is a monomial ideal then equality holds.*

*Remark 2.3.5.* Similar results appear in [35] and [7, Proposition 3.4].

**Example 2.3.6.** The inclusion $\rho_n(\operatorname{gr}(J)) \subset \operatorname{gr}(\rho_n(J))$ may be strict. Let

$$\begin{aligned} f &= \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{13}x_1x_3 + \alpha_{22}x_2^2 + \alpha_{23}x_2x_3 + \alpha_{33}x_3^2 \\ g &= \beta_{11}x_1^2 + \beta_{12}x_1x_2 + \beta_{13}x_1x_3 + \beta_{22}x_2^2 + \beta_{23}x_2x_3 + \beta_{33}x_3^2 \end{aligned}$$

where the set $\{\alpha_{ij}, \beta_{ij} | 1 \le i, j \le 3\}$ is algebraically independent[1] with respect to the prime subfield of $K$. In fact, $(f, g) = (h_1, h_2)$ where each truncation of the $h_i$'s are *generic forms* in the sense of [27, 30]. Then

$$\mathrm{gr}(f_1, f_2) = (x_1^2, x_1x_2, x_1x_3^2, x_2^4)$$

and

$$\rho_2(\mathrm{gr}(f_1, f_2)) = (x_1^2, x_1x_2, x_2^4) \subsetneq \mathrm{gr}(\rho_2(f_1, f_2)) = (x_1^2, x_1x_2, x_2^3).$$

**Corollary 2.3.7.** *If $J$ is an ideal of $R'$, then*

$$\rho_{n-1}(\mathrm{gr}(\rho_n(J))) \subset \mathrm{gr}(\rho_{n-1}(J)),$$

*thus we get a sequence*

$$\mathrm{gr}(\rho_0(J)) \xleftarrow{\rho_0} \mathrm{gr}(\rho_1(J)) \xleftarrow{\rho_1} \mathrm{gr}(\rho_2(J)) \xleftarrow{\rho_2} \mathrm{gr}(\rho_3(J)) \xleftarrow{\rho_3} \cdots$$

*Proof.* Applying Lemma 2.3.3, we have that

$$\rho_{n-1}(\mathrm{gr}(\rho_n(J))) \subset \mathrm{gr}(\rho_{n-1}(\rho_n(J))) = \mathrm{gr}(\rho_{n-1}(J)).$$

□

## 2.4 The ideal of infinitely recurring monomials

We have seen (for instance, from Example 2.3.6) that the truncated initial ideal $\mathrm{gr}(\rho_n(I))$ may contain monomials that are not in $\mathrm{gr}(I)$. Conversely, if $m \in \mathrm{gr}(I) \cap \mathcal{M}$ has maximal support greater than $n$, then obviously $m \notin \mathrm{gr}(\rho_n(I))$.

What about the monomials that lie in $\mathrm{gr}(\rho_n(I))$ for all sufficiently large $n$? Do they, by necessity, belong to $\mathrm{gr}(I)$?

**Definition 2.4.1.** If I is an ideal of $R'$, let

$$\widetilde{\mathrm{gr}(I)} = \bigcup_{i>0} \bigcap_{j>i} \mathrm{gr}(\rho_j(I))^e,$$

that is, $\widetilde{\mathrm{gr}(I)} \bigcap \mathcal{M}$ consists of those monomials that lie in $\mathrm{gr}(\rho_N(I))$ for all sufficiently large $N$.

**Lemma 2.4.2.** *If $I$ is an ideal of $R'$, then $\widetilde{\mathrm{gr}(I)}$ is a monomial ideal in $R'$.*

[1] For the definition of algebraic (in)dependence, see the discussion in [84] on “algebraische abhängigkeit” and “irreduzible Mengen”.

**Proposition 2.4.3.** *If* $I$ *is an ideal of* $R'$*, then* $\widetilde{\mathrm{gr}(I)} \supset \mathrm{gr}(I)$*. If* $I$ *is a monomial ideal, then equality holds.*

*Proof.* It is enough to verify that

$$\mathrm{gr}(I) \bigcap \mathcal{M} \subset \widetilde{\mathrm{gr}(I)} \bigcap \mathcal{M}.$$

Let $m \in \mathrm{gr}(I) \bigcap \mathcal{M}$, that is, $m = \mathrm{Lpp}(f)$ where $f \in J$. For large enough $n$ (more precisely, for $n >$ maxsupp$(m)$), Lemma 2.3.1 shows that $\mathrm{Lpp}(f) = \mathrm{Lpp}(\rho_n(f))$, hence $m \in \widetilde{\mathrm{gr}(J)}$.

If I is a monomial ideal then $I = \mathrm{gr}(I)$. Since $\rho_n(I)$ is a monomial ideal in $K[x_1, \ldots, x_n]$ for all $n$, we also have that

$$\rho_n(\mathrm{gr}(I)) = \rho_n(I) = \mathrm{gr}(\rho_n(I)).$$

It follows that $\widetilde{\mathrm{gr}(()} I) = \mathrm{gr}(I) = I$. □

## 2.5 Approximating the initial ideal of a locally finitely generated ideal

This section contains the main result of this article. The reader is reminded that $\mathcal{T}$ denotes the total-degree filtration on R and its K-sub-algebras.

### 2.5.1 Existence of a locally finite Gröbner basis

**Lemma 2.5.1.** *For a (proper) homogeneous ideal* $J \subset R'$*, the following are equivalent:*

*(i)* $\forall g : \dim_K \frac{J_g}{\sum_{j=1}^{g-1} R_j J_{g-j}} < \infty$,

*(ii) There exists a countable, homogeneous generating set* $S$ *of* $J$ *such that for all positive integers* $d$*, the set* $S_d = \{ s \in S \mid |s| = d \}$ *is finite,*

*(iii) There exists a countable generating set* $S$ *of* $J$ *such that for all positive integers* $d$*, the set* $S_d = \{ s \in S \mid |s| = d \}$ *is finite.*

A homogeneous ideal J fulfilling the conditions of Lemma 2.5.1 is called *locally finitely generated.* Countable subsets of $R'$ that contains only finitely many elements of a given total degree are called locally finite. Note that, in particular, finitely generated homogeneous ideals are locally finitely generated.

The following proposition is of vital importance for what is to follow. Although the result agrees with the intuition, and the naive idea of an inductive proof (assume that we have a finite, partial Gröbner basis up to degree $d$; add normal forms of the unprocessed generators of degree $d+1$, as well as normal forms of S-polynomials of degree $d+1$ of elements in the partial Gröbner basis; we have added a finite number of elements, so the partial Gröbner basis up to degree $d+1$ is finite) can be made to work, there are some tricky details, in particular with the proper definition of normal forms. The interested reader may consult [75].

**Proposition 2.5.2.** *If* $J$ *is locally finitely generated then so is* $\mathrm{gr}(J)$.

In what follows, $J$ will (unless otherwise stated) be a homogeneous, locally finitely generated ideal of $R'$. We will prove that the initial ideals $\mathrm{gr}(\rho_n(J))$, easily computable by standard Gröbner basis ttechniques, approximate $\mathrm{gr}(J)$. This result is summarized in Theorem 2.5.13.

### 2.5.2 A generating set of $\mathrm{gr}(J) \cap \mathcal{M}$

By Proposition 2.5.2 and Lemma 2.5.1 we can find a locally finite Gröbner basis $F = \{f_1, f_2, f_3, \dots\}$ of $J$, where $f_i$ is homogeneous, and there exists positive integers $\alpha(1) < \alpha(2) < \alpha(3) < \alpha(4) < \cdots$ such that for each total degree $d$, $|f_i| \leq d \iff i \leq \alpha(d)$.

We may assume that $F$ is minimal and reduced. Then, the set of leading monomials of $F$ is a minimal generating set for $\mathrm{gr}(J)$:

**Definition 2.5.3.** Put $m_i = \mathrm{Lpp}(f_i)$ for all $i$ and let $B = \{m_i\}$. Then we have that $\langle B \rangle = \mathrm{gr}(J) \cap \mathcal{M}$ and $\langle B \rangle_{R'} = \mathrm{gr}(J)$, where we use the notational apparatus of [75]: $\langle B \rangle$ denotes the semi-group ideal generated by $B$ in $\mathcal{M}$, and $\langle B \rangle_{R'}$ denotes the (monomial) ideal generated by $B$ in $R'$.

As a notational convenience, we denote, for any $d$, by $B_d$ and by $B_{\leq d}$ the sets

$$B_d = \{\, m \in B \,|\, |m| = d \,\} = \{m_{\alpha(d-1)+1}, \dots, m_{\alpha(d)}\}$$
$$B_{\leq d} = \mathcal{T}^d B = \{\, m \in B \,|\, |m| \leq d \,\} = \{m_1, m_2, m_3, \dots, m_{\alpha(d)}\}$$

**Lemma 2.5.4.** $\mathcal{T}^d \mathrm{gr}(J) = \mathcal{T}^d \langle B_{\leq d} \rangle_{R'}$. *Furthermore, the* $K$ *vector space* $\frac{\mathrm{gr}(J)_d}{\sum_{j=1}^{d-1} R'_j \,\mathrm{gr}(J)_{d-j}}$ *is minimally generated by the images of elements in* $B_d$.

### 2.5.3 The necessary number of active variables

It will be of great importance to keep track of how many "active" variables are needed up to a given degree. The following definition makes this notion more precise.

When approximating $\mathcal{T}^d \mathrm{gr}(J)$ with $\mathcal{T}^d \mathrm{gr}(\rho_n(J))$, we certainly need at least as many active variables, that is, at least as large $n$, as when approximating with

$\mathcal{T}^d\rho_n(\mathrm{gr}(J))$. The latter quantity, that is, the least $N(d)$ such that $\mathcal{T}^d\rho_n(\mathrm{gr}(J)) = \mathcal{T}^d\,\mathrm{gr}(J)$ whenever $n \geq N(d)$, is of course determined by $B$.

**Definition 2.5.5.** The "the necessary number of active variables up to degree $d$", $N(d)$, is defined as

$$N(d) = \max\left(\{\,\mathrm{maxsupp}(m) | m \in B_{\leq d}\}\right).$$

### 2.5.4 *Restricting* $B$

It is clear that

$$\rho_0(B) \subset \rho_1(B) \subset \rho_2(B) \subset \rho_3(B) \subset \cdots \tag{2.3}$$

For an infinite $B$, (2.3) will not stabilize. However:

**Lemma 2.5.6.** *For a fixed* $d$*, the chain of inclusions*

$$\begin{aligned} \rho_0(B_{\leq d}) \subset \rho_1(B_{\leq d}) \subset \rho_2(B_{\leq d}) \subset \rho_3(B_{\leq d}) \subset \cdots \\ \subset \rho_{N(d)-1}(B_{\leq d}) \subset \rho_{N(d)}(B_{\leq d}) = \rho_{N(d)+1}(B_{\leq d}) = \rho_{N(d)+2}(B_{\leq d}) = \cdots \end{aligned} \tag{2.4}$$

*stabilizes at* $N(d)$.

We now use (2.3) and (2.4) to construct ascending chains of ideals in $R'$: from (2.3) we get the (non-stabilizing) sequence

$$\rho_0(\mathrm{gr}(J))^e \subset \rho_1(\mathrm{gr}(J))^e \subset \rho_2(\mathrm{gr}(J))^e \subset \rho_3(\mathrm{gr}(J))^e \subset \cdots \tag{2.5}$$

and from (2.4) the stabilizing sequence

$$\begin{aligned} &\mathcal{T}^d\rho_0(\mathrm{gr}(J))^e \subset \mathcal{T}^d\rho_1(\mathrm{gr}(J))^e \subset \mathcal{T}^d\rho_2(\mathrm{gr}(J))^e \subset \cdots \\ &\qquad \cdots \subset \mathcal{T}^d\rho_{N(d)-1}(\mathrm{gr}(J))^e \subset\subset \mathcal{T}^d\rho_{N(d)}(\mathrm{gr}(J))^e = \\ &\qquad\qquad = \mathcal{T}^d\rho_{N(d)+1}(\mathrm{gr}(J))^e = \mathcal{T}^d\rho_{N(d)+2}(\mathrm{gr}(J))^e = \cdots \end{aligned} \tag{2.6}$$

**Lemma 2.5.7.** *The stable value* $\mathcal{T}^d\rho_{N(d)}(\mathrm{gr}(J))^e$ *is equal to* $\mathcal{T}^d\,\mathrm{gr}(J)$.

### 2.5.5 *Relating the truncated initial ideals and the initial ideal*

We know from Corollary 2.3.4 that for all $n$,

$$\rho_n(\mathrm{gr}(J))^e \subset \mathrm{gr}(\rho_n(J))^e,$$

in particular,

$$\mathcal{T}^d\rho_n(\mathrm{gr}(J))^e \subset \mathcal{T}^d\,\mathrm{gr}(\rho_n(J))^e.$$

For $n \geq N(d)$ we get that

$$\mathcal{T}^d \operatorname{gr}(J) = \mathcal{T}^d \rho_n(\operatorname{gr}(J))^e \subset \mathcal{T}^d \operatorname{gr}(\rho_n(J))^e.$$

Combining this with previous results, we can draw the following diagram:

$$\begin{array}{ccccc}
\mathcal{T}^d \operatorname{gr}(\rho_{N(d)-1}(J))^e & & \mathcal{T}^d \operatorname{gr}(\rho_{N(d)}(J))^e & & \mathcal{T}^d \operatorname{gr}(\rho_{N(d)+1}(J))^e \\
\bigcup & & \bigcup & & \bigcup \\
\mathcal{T}^d \rho_{N(d)-1}(\operatorname{gr}(J))^e & \subset & \mathcal{T}^d \rho_{N(d)}(\operatorname{gr}(J))^e & = & \mathcal{T}^d \rho_{N(d)+1}(\operatorname{gr}(J))^e \\
\bigcap & & \| & & \| \\
\mathcal{T}^d \operatorname{gr}(J) & & \mathcal{T}^d \operatorname{gr}(J) & & \mathcal{T}^d \operatorname{gr}(J)
\end{array} \tag{2.7}$$

*Remark 2.5.8.* It is not necessarily the case that

$$\operatorname{gr}(\rho_n(J))^e \subset \operatorname{gr}(\rho_{n+1}(J))^e,$$

nor that

$$\mathcal{T}^d \operatorname{gr}(\rho_n(J))^e \subset \mathcal{T}^d \operatorname{gr}(\rho_{n+1}(J))^e.$$

In most cases, the inclusion

$$\mathcal{T}^d \operatorname{gr}(J) = \mathcal{T}^d \rho_{N(g)}(\operatorname{gr}(J))^e \subset \mathcal{T}^d \operatorname{gr}(\rho_{N(d)}(J))^e$$

will be strict. Thus, we may perform the $K$-vector space decomposition

$$\mathcal{T}^d \operatorname{gr}(\rho_{N(d)}(J))^e = \mathcal{T}^d \operatorname{gr}(J) \oplus Q_{N(d)}$$

where, in general, $Q_{N(d)}$ is non-zero. In fact, we can make this decomposition for any $n \geq N(d)$, obtaining a sequence of $K$-vector spaces $Q_n$. Our next aim is to prove that there exists an integer $\hat{N}(d)$, “the sufficient number of active variables up to degree d”, such that $n \geq \hat{N}(d) \implies Q_n = 0$. For such $n$'s we will then have that

$$\mathcal{T}^d \operatorname{gr}(\rho_n(J))^e = \mathcal{T}^d \rho_n(\operatorname{gr}(J))^e = \mathcal{T}^d \operatorname{gr}(J).$$

We can complete diagram (2.7) and get

$$
\begin{array}{rcccl}
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{\hat{N}(d)+1}(J))^{e} & = & \mathcal{T}^{d}\rho_{\hat{N}(d)+1}(\mathrm{gr}(J))^{e} & = & \mathcal{T}^{d}\,\mathrm{gr}(J)\\
 & & \| & & \\
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{\hat{N}(d)}(J))^{e} & = & \mathcal{T}^{d}\rho_{\hat{N}(d)}(\mathrm{gr}(J))^{e} & = & \mathcal{T}^{d}\,\mathrm{gr}(J)\\
 & & \cup & & \\
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{\hat{N}(d)-1}(J))^{e} & \supset & \mathcal{T}^{d}\rho_{\hat{N}(d)-1}(\mathrm{gr}(J))^{e} & = & \mathcal{T}^{d}\,\mathrm{gr}(J)\\
 & & \cup & & \\
 & & \vdots & & \\
 & & \cup & & \\
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{N(d)+1}(J))^{e} & \supset & \mathcal{T}^{d}\rho_{N(d)+1}(\mathrm{gr}(J))^{e} & = & \mathcal{T}^{d}\,\mathrm{gr}(J)\\
 & & \cup & & \\
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{N(d)}(J))^{e} & \supset & \mathcal{T}^{d}\rho_{N(d)}(\mathrm{gr}(J))^{e} & = & \mathcal{T}^{d}\,\mathrm{gr}(J)\\
 & & \cup & & \\
\mathcal{T}^{d}\,\mathrm{gr}(\rho_{N(d)-1}(J))^{e} & \supset & \mathcal{T}^{d}\rho_{N(d)-1}(\mathrm{gr}(J))^{e} & \subset & \mathcal{T}^{d}\,\mathrm{gr}(J)
\end{array}
\tag{2.8}
$$

### 2.5.6 Reducing S-polynomials

It is proved in [75] that every *S-polynomial* of elements of the chosen locally finite Gröbner basis $F$ reduce to zero with respect to $F$. That is to say, each such S-polynomial can be expressed as an *admissible combination* of elements in $F$. We now fix a choice of such admissible combinations.

For any $1 \leq i < j$, choose $a_{i,j,1}, a_{i,j,2}, \ldots a_{i,j,\alpha(|f_j|)} \in R'$ such that

$$
S(f_i, f_j) = \sum_{k=1}^{\alpha(|f_j|)} a_{i,j,k} f_k \ , \ \forall k : \mathrm{Lpp}(S(f_i, f_j)) \geq \mathrm{Lpp}(a_{i,j,k})\,\mathrm{Lpp}(f_k)
$$

(the right-hand side is an admissible combination). Furthermore, we can also ensure that no $a_{i,j,k}$ have higher total degree than $S(f_i, f_j)$.

Put

$$
A = \{\, a_{i,j,k} | 1 \leq i < j;\ k \leq \alpha(|f_j|)\,\},
$$

and define, for any total degree $d$,

$$
A_{\leq d} = \{\, a_{i,j,k} \in A \,|\, |S(f_i, f_j)| \leq d \,\}.
$$

That is, $A_{\leq d}$ consists of those $a_{i,j,k}$ that are involved in reducing those S-polynomials of elements in $F$ that have total degree $\leq d$. Since $F$ is locally finite there are only finitely many such S-polynomials. We conclude that $A_{\leq d}$ is finite; this will be of utmost importance.

**Definition 2.5.9.** Let "the sufficient number of active variables up to degree $d$" be defined as

$$\hat{N}(d) = \max\left(N(d), \max\{\,\mathrm{maxsupp}(\mathrm{Lpp}(a)) \,|\, a \in A_{\leq d}\}\right).$$

We remark that this number unfortunately depends not only on $J$ but also on the choice of $A$.

*Remark 2.5.10.* By construction, we have that $N(d) \leq \hat{N}(d)$.

**Lemma 2.5.11.** *If* $P, Q \in R'$, *and if*

$$n \geq \max\left(\mathrm{maxsupp}(\mathrm{Lpp}(P)), \mathrm{maxsupp}(\mathrm{Lpp}(Q))\right)$$

*then*

$$\rho_n(S(P,Q)) = S(\rho_n(P), \rho_n(Q)).$$

*Proof.* Assume, to simplify things, that $P$ and $Q$ are monic, with leading power products $p$ and $q$ respectively, and that the least common multiple of $p$ and $q$ is $m$. Then $S(P,Q) = \frac{m}{p}P - \frac{m}{q}Q$, and

$$\rho_n(S(P,Q)) = \frac{m}{p}\rho_n(P) - \frac{m}{q}\rho_n(Q).$$

On the other hand, $\rho_n(P) = p$ and $\rho_n(Q) = q$, hence

$$S(\rho_n(P), \rho_n(Q)) = \frac{m}{p}P - \frac{m}{q}Q.$$

□

### 2.5.7 Truncating admissible combinations

Fix a total degree $d$. If $1 \leq i < j \leq \alpha(d)$ then $|f_i|, |f_j|, |S(f_i, f_j)| \leq d$ and

$$S(f_i, f_j) = \sum_{k=1}^{\alpha(|f_j|)} a_{i,j,k} f_k \tag{2.9}$$

where the right-hand side is an admissible combination of elements in $F_{\leq d}$. For any $n$, it is clear that

$$\rho_n(S(f_i, f_j)) = \sum_{k=1}^{\alpha(|f_j|)} \rho_n(a_{i,j,k})\rho_n(f_k).$$

For $n \geq N(d)$ we have that

$$\rho_n(S(f_i, f_j)) = S(\rho_n(f_i), \rho_n(f_j)),$$

by Lemma 2.5.11. Finally, if $n \geq \hat{N}(d)$ then

$$\begin{aligned} \mathrm{Lpp}(S(\rho_n(f_i), \rho_n(f_j))) &= \mathrm{Lpp}(S(f_i, f_j)) \\ \mathrm{Lpp}(\rho_n(f_k)) &= \mathrm{Lpp}(f_k) \\ \mathrm{Lpp}(\rho_n(a_{i,j,k})) &= \mathrm{Lpp}(a_{i,j,k}) \end{aligned}$$

hence for such $n$, every admissible combination such as (2.9), reducing to zero an S-polynomial of elements of $F$ with total degree $\leq d$, restricts to an admissible combination in $K[x_1, \ldots, x_n]$.

### 2.5.8 The main theorem

**Lemma 2.5.12.** *Let $I$ be a homogeneous ideal in $K[x_1, \ldots, x_n]$ generated by a finite, homogeneous set $G$. Let $t$ be a positive integer, and suppose that all S-polynomials of elements in $G$, except those that have total degree higher than $t$, reduce to zero with respect to $G$. Then, for each $g \leq t$,*

$$(\mathrm{gr}(I))_g = \langle \mathrm{in}(F) \rangle_{R'_g}.$$

*Proof.* The result is well-known; there is a simple proof of it in [75]. □

This result can immediately be generalized to the case of a locally finitely generated ideal, simply by applying the corollary to the sub-ideal generated by those (finitely many) elements of the locally finite generating set that have total degree $\leq t$.

From the discussion above, we know that $\rho_n(F_{\leq d})$ is such a "partial Gröbner basis" for $\rho_n(J) \subset K[x_1, \ldots, x_n]$, (when $n \geq \hat{N}(d)$), and hence we conclude that

$$\mathcal{T}^d \mathrm{gr}(\rho_n(J)) \cap \mathcal{M} = \mathcal{T}^d \mathcal{M}^n B_{\leq d} \tag{2.10}$$

We then get that

$$\mathcal{T}^d \mathrm{gr}(\rho_n(J))^e \cap \mathcal{M} = \mathcal{T}^d \mathcal{M} B_{\leq d} \tag{2.11}$$

and that

$$\mathcal{T}^d \mathrm{gr}(\rho_n(J))^e = \mathcal{T}^d \mathrm{gr}(J) \tag{2.12}$$

This is the desired result! It implies immediately that the K-vector space $Q_n$, defined previously, is zero. We summarize our results in the following theorem:

**Theorem 2.5.13 (Degree-wise approximation of initial ideals).** *If $J$ is a locally finitely generated ideal in $R'$, then for all total degrees $d$ we have that*

$$L(d,n) := \mathcal{T}^d \rho_n(\mathrm{gr}(J))^e \subset \mathcal{T}^d \mathrm{gr}(\rho_n(J))^e =: R(d,n) \tag{2.13}$$

*Furthermore, there exists integers $N(d)$, which we call "the necessary number of active variables up to degree $d$", and integers $\hat{N}(d)$, which we call "the sufficient number of active variables up to degree $d$", such that:*

*(i) If $n < N(d)$ then*

$$L(d,n) \subsetneq \mathcal{T}^d \mathrm{gr}(J) \tag{2.14}$$
$$R(d,n) \not\supset \mathcal{T}^d \mathrm{gr}(J) \tag{2.15}$$

*(ii) If $N(d) \leq n < \hat{N}(d)$ then*

$$L(d,n) = \mathcal{T}^d \mathrm{gr}(J) \tag{2.16}$$
$$R(d,n) \supset \mathcal{T}^d \mathrm{gr}(J) \tag{2.17}$$

*(iii) If $\hat{N}(d) \leq n$ then*

$$L(d,n) = \mathcal{T}^d \mathrm{gr}(J) \tag{2.18}$$
$$R(d,n) = \mathcal{T}^d \mathrm{gr}(J) \tag{2.19}$$

### 2.5.9 Some consequences of the approximation theorem

**Corollary 2.5.14.** *The following are equivalent:*

*(i)* $\mathrm{gr}(J)$ *is finitely generated,*

*(ii)* $\mathrm{gr}(\rho_n(J))^e$ *stabilize when $n$ tends to infinity.*

*Furthermore, if the equivalent conditions hold, then $J$ is finitely generated.*

*Proof.* If $\mathrm{gr}(J)$ is finitely generated, it is generated by a finite set of monomials. Therefore, there exists an integer $N$ such that all these monomials are contained in $\mathcal{M}^N$. Hence, the semi-group $\mathrm{gr}(J) \cap \mathcal{M}$ is generated in $\mathcal{M}^N$, and $\mathrm{gr}(\rho_n(J))^e) = \mathrm{gr}(J)$ whenever $n \geq N$.

Conversely, if there exists an integer $N$ such that

$$\mathrm{gr}(\rho_n(J))^e = \mathrm{gr}(\rho_k(J))^e$$

whenever $n, k \geq N$, then by Theorem 2.5.13 this common value is $\mathrm{gr}(J)$. We conclude that $\mathrm{gr}(J) \cap \mathcal{M}$ is generated in $\mathcal{M}^N$ and consequently that $\mathrm{gr}(J)$ is generated in $K[x_1, \ldots, x_n]$. It follows that $\mathrm{gr}(J)$ is finitely generated.

The last assertion is clear: a Gröbner basis is also a generating set. □

**Corollary 2.5.15.** $\widetilde{\mathrm{gr}(J)} = \mathrm{gr}(J)$.

*Proof.* By Proposition 2.4.3, the inclusion $\supset$ holds. Now let $m \in \widetilde{\mathrm{gr}(J)} \cap \mathcal{M}$, so that $m \in \mathrm{gr}(\rho_n(J)) \cap \mathcal{M}$ for all sufficiently large $n$. Denote by $g$ the degree of $m$. We know from Theorem 2.5.13 that

$$\mathrm{gr}(J)_g \cap \mathcal{M} = \mathrm{gr}(\rho_n(J)) \cap \mathcal{M}$$

for all sufficiently large $n$. Clearly $m \in \mathrm{gr}(\rho_n(J))$ for all such $n$, hence $m \in \mathrm{gr}(J)$. □

**Question 2.5.16.** *For arbitrary ideals* $I \subset R'$, *is it true that* $\widetilde{\mathrm{gr}(I)} = \mathrm{gr}(I)$ *?*

**Corollary 2.5.17.** $\mathrm{gr}(J)$ *is completely determined by the ideals* $\mathrm{gr}(\rho_n(J))^e$, *and hence by the ideals* $\mathrm{gr}(\rho_n(J))$. *Therefore,* $\mathrm{gr}(J)$ *is determined by the ideals* $\rho_n(J)$.

**Question 2.5.18.** *Is* $J$ *itself determined by the ideals* $\rho_n(J)$*?*

Note that this question has a negative answer for non-locally finitely generated ideals: if

$$I = (x_1, x_2, x_3, \dots)$$

whereas

$$I' = I + (x_1 + x_2 + x_3 + \cdots)$$

then $I \neq I'$ but

$$\rho_n(I) = \rho_n(I') = (x_1, \dots, x_n)$$

for all $n$.

The author has recently proved [79] that the answer to Question 2.5.18 is "yes": locally finitely generated ideals are determined by their truncations. The idea of the proof is to topologize $R'$ by the separated filtration given by the kernels of the truncation homomorphisms, and then show that in this topology, locally finitely generated ideals are closed.

### *The mysterious* $\hat{N}(d)$

Theorem 2.5.13 is unsatisfactory in one aspect: it does not really tell us how to compute $\mathcal{T}^d \mathrm{gr}(J)$ from the initial ideals of the restricted ideals $\rho_n(J)$, since it does not provide any hints as how to find the number $\hat{N}(d)$. We can, of course, use the methods of [75] to find it, but that involves calculating $\mathcal{T}^d \mathrm{gr}(J)$ directly.

Instead, one would like to perform calculations of $\mathrm{gr}(\rho_n(J))$ with increasing $n$, and from inspecting the results determine when the "stable value at degree $d$" has been reached. Ideally, we should be able to compute $\mathcal{T}^d \mathrm{gr}(\rho_n(J))^e$ for successively larger values of $n$, and then, when this sequence seems to have reached its stable value, because it has not changed for $k$ consecutive values for $n$, conclude that we have indeed reached the necessary number of active variables.

**Question 2.5.19.** *Does there, for each homogeneous, locally finitely generated ideal* $I \subset R'$*, exist a* $k$*, independent of* $d$*, such that*

$$\mathcal{T}^{d}\,\mathrm{gr}(\rho_n(I))^e = \mathcal{T}^{d}\,\mathrm{gr}(\rho_{n+1}(I))^e = \cdots = \mathcal{T}^{d}\,\mathrm{gr}(\rho_{n+k}(I))^e$$

*implies that* $\mathcal{T}^{d}\,\mathrm{gr}(\rho_n(I))^e = \mathcal{T}^{d}\,\mathrm{gr}(I)$*?*

If this fails, one would be interested in the answer to the following question:

**Question 2.5.20.** *Given* $J$ *and* $d$*, is there a faster way of computing* $\hat{N}(d)$ *than by calculating a partial Gröbner basis for* $J$ *up to degree* $d$*?*

Partial results, such as for generic ideals, or for a restricted set of admissible orders, would be interesting, should the general problem be hard to solve.

## 2.6 The lex-initial ideal of a 2-4 generic ideal

In this section, we will calculate the initial ideal (with respect to the lexicographic order) of the generic ideal $I$ generated by a generic quadratic form and a generic form of degree 4. By "generic ideal", we mean, as in [30, 27], that not only are the generators generic, but they are independent in the sense that the union of their sets of coefficients is algebraically independent.

Tables 2.1 and 2.2 shows the initial ideals of the restricted ideals $\rho_2(I)$ to $\rho_7(I)$. These restricted ideals are of course generic ideals in their corresponding polynomial rings. In the interest of brevity, we show only $\mathcal{T}^{9}\,\mathrm{gr}(\rho_n(I))$ which means that only the first three initial ideals are showed in their entirety.

From these tables, we see that it is very plausible that

$$\mathcal{T}^{7}\,\mathrm{gr}(I) = (x_1^2,\ x_1x_2^3,\ x_1x_2^2x_3^2,\ x_1x_2^2x_3x_4^2,\ x_1x_2^2x_3x_4x_5^2,\ x_1x_2^2x_3x_4x_5x_6).$$

By considering also the restricted ideals with as many as 11 variables, one can be rather certain that the minimal monomial generators of $\mathrm{gr}(I)$ of degree 8 are

$$x_2^8,\ x_1x_2x_3^6,\ x_1x_2^2x_3x_4x_5x_7x_{10},\ x_1x_2^2x_3x_4x_5x_7x_9,\\ x_1x_2^2x_3x_4x_5x_7x_8,\ x_1x_2^2x_3x_4x_5x_7^2.$$

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| Degree | $\mathrm{gr}(\rho_2(I))$ | $\mathrm{gr}(\rho_3(I))$ | $\mathrm{gr}(\rho_4(I))$ |
|---|---|---|---|
| 2 | $x_1^2$ | $x_1^2$ | $x_1^2$ |
| 4 | $x_1x_2^3$ | $x_1x_2^3$ | $x_1x_2^3$ |
| 5 | $x_2^5$ | $x_1x_2^2x_3^2$ | $x_1x_2^2x_3^2$ |
| 6 | | $x_1x_2x_3^4$ | $x_1x_2^2x_3x_4^2$ |
| 7 | | | $x_1x_2^2x_4^4$ |
| 7 | | $x_1x_3^6$ | $x_1x_2x_3^5$ |
| 8 | | | |
| 8 | | | |
| 8 | | | |
| 8 | | | |
| 8 | | | $x_1x_2x_3^4x_4^2$ |
| 8 | | $x_2^8$ | $x_2^8$ |
| 9 | | | $x_1x_2x_3^3x_4^4$ |

***Tab. 2.1:*** *Initial ideals of truncations of the generic ideal generated by a quadratic form and a form of degree 4, 2 to 4 variables*

| Degree | $gr(\rho_5(I))$ | $gr(\rho_6(I))$ | $gr(\rho_7(I))$ |
|---|---|---|---|
| 2 | $x_1^2$ | $x_1^2$ | $x_1^2$ |
| 4 | $x_1x_2^3$ | $x_1x_2^3$ | $x_1x_2^3$ |
| 5 | $x_1x_2^2x_3^2$ | $x_1x_2^2x_3^2$ | $x_1x_2^2x_3^2$ |
| 6 | $x_1x_2^2x_3x_4^2$ | $x_1x_2^2x_3x_4^2$ | $x_1x_2^2x_3x_4^2$ |
| 7 | $x_1x_2^2x_3x_4x_5^2$ | $x_1x_2^2x_3x_4x_5^2$ | $x_1x_2^2x_3x_4x_5^2$ |
| 7 | $x_1x_2^2x_3x_5^3$ | $x_1x_2^2x_3x_4x_5x_6$ | $x_1x_2^2x_3x_4x_5x_6$ |
| 8 | | $x_1x_2^2x_3x_4x_6^3$ | $x_1x_2^2x_3x_4x_5x_7^2$ |
| 8 | $x_1x_2^2x_4^5$ | $x_1x_2^2x_3x_5^4$ | $x_1x_2^2x_3x_4x_6^3$ |
| 8 | $x_1x_2^2x_4^4x_5$ | $x_1x_2^2x_3x_5^3x_6$ | $x_1x_2^2x_3x_4x_6^2x_7$ |
| 8 | $x_1x_2^2x_4^3x_5^2$ | $x_1x_2^2x_3x_5^2x_6^2$ | $x_1x_2^2x_3x_4x_6x_7^2$ |
| 8 | $x_1x_2x_3^6$ | $x_1x_2x_3^6$ | $x_1x_2x_3^6$ |
| 8 | $x_2^8$ | $x_2^8$ | $x_2^8$ |
| 9 | | | $x_1x_2^2x_3x_4x_7^4$ |
| 9 | | | $x_1x_2^2x_3x_5^5$ |
| 9 | | | $x_1x_2^2x_3x_5^4x_6$ |
| 9 | | | $x_1x_2^2x_3x_5^4x_7$ |
| 9 | | $x_1x_2^2x_3x_5x_6^4$ | $x_1x_2^2x_3x_5^3x_6^2$ |
| 9 | | $x_1x_2^2x_3x_6^5$ | $x_1x_2^2x_3x_5^3x_6x_7$ |
| 9 | | $x_1x_2^2x_4^6$ | $x_1x_2^2x_3x_5^3x_7^2$ |
| 9 | | $x_1x_2^2x_4^5x_5$ | $x_1x_2^2x_3x_5^2x_6^3$ |
| 9 | $x_1x_2^2x_4^2x_5^4$ | $x_1x_2^2x_4^5x_6$ | $x_1x_2^2x_4^6$ |
| 9 | $x_1x_2^2x_4x_5^5$ | $x_1x_2^2x_4^4x_5^2$ | $x_1x_2^2x_4^5x_5$ |
| 9 | $x_1x_2x_3^5x_4^2$ | $x_1x_2x_3^5x_4^2$ | $x_1x_2x_3^5x_4^2$ |

***Tab. 2.2:*** *Initial ideals of truncations of the generic ideal generated by a quadratic form and a form of degree 4, 5 to 7 variables*

# 3. REVERSE LEXICOGRAPHIC INITIAL IDEALS OF GENERIC IDEALS ARE FINITELY GENERATED

A abridged version will appear in *Gröbner Bases and Applications*, (Proceedings of the Conference 33 years of Gröbner Bases), volume 251 of London Mathematical Society Lecture Notes Series, published by Cambridge University Press.


*Abstract*

This article generalizes the well-known notion of *generic forms* to the algebra $R'$, introduced in [75]. For the total degree, then reverse lexicographic order, we prove that the initial ideal of an ideal generated by finitely many generic forms (in countably infinitely many variables) is finitely generated. This contrasts to the lexicographic order, for which initial ideals of generic ideals in general are non-finitely generated.

We use the approximation methods developed in [76], together with the results of Moreno in [55] on "ordinary" initial ideals of reverse lexicographic initial ideals of generic ideals, to prove that a minimal generating set of the initial ideal of an ideal generated by $k$ generic forms is contained in the semi-group $\mathcal{M}^k$; hence, it is finite.

As a generalization of this result, we prove that what we call "pure generic" ideals in an non-noetherian overring of a polynomial ring on two groups of variables, have initial ideals (with respect to a "twisted" product order of degrevlex on the two groups) that are finitely generated.

The natural question, "is the reverse lexicographic initial ideal of an homogeneous, finitely generated ideal in $R'$ finitely generated" is posed, but not answered; we do, however, point out one direction of investigation that might provide the answer: namely to view such an ideal as the "specialization" of a generic ideal.


## 3.1 Introduction

In this article, we study the initial ideals of generic and "almost generic" ideals with respect to the (total degree, then) reverse lexicographic term order. For a generic ideal $I \subset K[x_1, \ldots, x_n]$, generated by $r \leq n$ forms, there is a well-known conjecture on how $\mathrm{gr}(I)$ looks like. In particular, $\mathrm{gr}(I)$ is minimally generated in

$K[x_1, \ldots, x_r]$. We interpret this result in the setting of the ring $R'$, introduced in [75]: this ring, which is a proper subring on the power series ring on countably many variables, and which properly contains the polynomial ring on the same set of indeterminates, is the habitat of "generic forms in countably many indeterminates". In the non-noetherian ring $R'$, finitely generated, homogeneous ideals need not have finitely generated initial ideals; in fact, there are many finitely generated, generic ideals that have non-finitely generated initial ideals, with respect to the pure lexicographic term order. However, we show that the result above implies that finitely generated, generic ideals in $R'$ have finitely generated initial ideals with respect to the reverse lexicographic term order. The key property of the degrevlex order that we use is the fact that the forming of initial ideals with respect to this order commutes with the truncation homomorphisms $\rho_n$, so that $\mathrm{gr}(\rho_n(I)) = \rho_n(\mathrm{gr}(I))$, whereas for arbitrary term orderings we only have an inclusion.

We also study variants of generic ideals, where the coefficients of the monomials of the forms lie not in the field, but in some other polynomial ring, which is mapped onto the ground field by a *specialization* map. We call such ideals *pure generic ideals*. At first, we study them in the polynomial ring $K[y_1, \ldots, y_{t_n}; x_1, \ldots, x_n]$, where we show that their initial ideals, with respect to the "twisted" product order of degrevlex on the two groups of variables, is minimally generated in $K[y_1, \ldots, y_{t_r}; x_1, \ldots, x_r]$, if the pure generic ideal is generated by $r$ pure generic forms.

This construction can be generalized to the ring $K[Y][[X]]'$. We prove similar results on the initial ideals of pure generic ideals. In particular, we show that they are finitely generated.

Finally, we study *specialization maps* from this ring to $R'$, that is, maps which fix the X-variables and map $K[Y]$ onto $K$. Since every finitely generated, homogeneous ideal in $R'$ may be regarded as the specialization of a generic ideal, it is natural to ask if the initial ideal (with respect to the reverse lexicographic term order) of a finitely generated, homogeneous ideal in $R'$ is finitely generated. We are unable to answer this question, but we present some ideas that might be used to tackle it.

## 3.2 Preliminaries

The rings, algebras, semi-groups and other devices used below are defined in greater detail in [75, 76], to which we refer the reader.

Let $K$ be a field, and let $Q$ be its prime field. For any positive integer $n$, we denote by $\mathcal{M}^n$ the free commutative semigroup on the letters $\{x_1, \ldots, x_n\}$, and by $\mathcal{M}^n_d$ the subset of elements of total degree $d$. Since the polynomial ring $K[x_1, \ldots, x_n]$ is the monoid ring of $\mathcal{M}^n$ over $K$, we can identify it with the set of all finitely supported maps from $\mathcal{M}^n$ to $K$. For an arbitrary element $h \in K[x_1, \ldots, x_n]$, we denote by $\mathrm{Coeff}(m, h)$ the value of the corresponding map at $m \in \mathcal{M}^n$, and by $\mathrm{Mon}(h)$ the support of the map.

We mean by a *form* of degree $|f| = d$ a homogeneous element. This element is said to be a *generic form* if, in addition, the set of its coefficients, that is, the set $\{\mathrm{Coeff}(m, f) \mid m \in \mathrm{Mon}(f)\}$ is algebraically independent over Q, if no two coefficients are equal, and if every monomial of appropriate total degree occur in the set of monomials: $\mathrm{Mon}(f) = \mathcal{M}^n_d$. An ideal I of $K[x_1, \ldots, x_n]$ is said to be *generic* if we can find a (finite) generating set, whose members are generic forms, and furthermore the union of the sets of coefficients of the generators is algebraically independent over Q; we also demand that no two occuring coefficients are equal.

These concepts are well-known and well-studied by algebraists ([30, 27, 84]). We now generalize them to (countably) infinitely many variables. For this purpose, we first introduce $R = K[[x_1, x_2, x_3, \ldots]]$, the power series ring on countably many variables, and then define the $K$-algebra $R'$ as the sub-algebra of $R$ that is generated by all homogeneous elements. We denote by $\mathcal{M}$ the free commutative monoid on the $x_i$'s (in other words, the direct limit of the $\mathcal{M}^n$'s) and by $\mathcal{M}_d$ the subset of all elements of degree $d$. Then, elements of $R$ may be viewed as maps from $\mathcal{M}$ to $K$, and we can define $\mathrm{Coeff}(m, h)$ and $\mathrm{Mon}(h)$ analogously to the polynomial case. We remark that similar rings have been studied extensively in the litterature; see for instance [64, 65, 66, 69, 67, 68], [50], [15].

We mean by a *form in* $R'$ a homogeneous element $f$ in $R'$. A *generic form in* $R'$, is a form $f$ in $R'$ such that $\{\mathrm{Coeff}(m, f) \mid m \in \mathrm{Mon}(f)\}$ is algebraically independent over Q, such that no two coefficients occuring are equal, and such that $\mathrm{Mon}(f) = \mathcal{M}_{|f|}$. By a *generic ideal in* $R'$ we mean an ideal I for which a finite set of generators, which are generalized forms, can be found, such that the union of the sets of coefficients for the generators is algebraically independent over Q, and such that no two coefficients occuring are equal. In particular, such an ideal is homogeneous and finitely generated.

We assume that $K$ contains infinitely many elements that are transcendental over Q, and algebraically independent over Q.

In this article, except where otherwise stated, $>$ will denote the total-degree, then reverse lexicographic order on the semi-group $\mathcal{M}$ of monomials in the variables $x_1, x_2, x_3, \ldots$, as well as its restriction to the subsemigroups $\mathcal{M}^n$. It is enough to define $>$ on each $\mathcal{M}^n$, where

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} > x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$ or the total degrees are equal but

$$\exists r \in \{1, \ldots, n\} : (\alpha_r < \beta_r) \wedge (i > r \implies \alpha_i = \beta_i).$$

We say that $>$ is an *admissible order* on $\mathcal{M}$, by which we mean that it is a total order with $1$ as the smallest element and such that

$$x_i > x_j \iff i < j, \quad p > q \implies pt > pq, \quad p, q, t \in \mathcal{M}.$$

It was showed in [75] that if $f \in R'$ and $>$ is an admissible order on $M$, then $\mathrm{Mon}(f) \subset \mathcal{M}$ has a maximal element (with respect to $>$) $\mathrm{Lpp}(f)$, which we call the *leading power product* of $f$. Therefore, we can associate to any ideal $I$ in $R'$ its *initial ideal* $\mathrm{gr}(I)$, the monomial ideal generated by all leading power products of elements in $I$. It was also showed that if $I$ is locally finitely generated, that is, homogeneous and posesses a homogeneous generating set with only finitely many elements of any given total degree, then the initial ideal share that property. In order to show this, a Gröbner basis theory for locally finitely generated ideals in $R'$ was developed. Since the polynomial rings $K[x_1, \ldots, x_n]$ are embedded in $R'$, this theory extends the classical theory pioneered by Buchberger [18, 19, 21, 22] (see also [11, 72, 59]). In fact, most of the well-known results carry over to this case, and the proofs are either trivial modifications of the ordinary proofs, or reductions to the polynomial ring case. There are however some dissimilarites, due to the fact that $R'$ is non-noetherian.

If $n$ is any positive integer, denote by $B_n$ the ideal generated in $R$ by all power series in $K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]]$ with zero constant term. Then the *$n$'th truncation homomorphism* is defined by

$$\rho_n : R \twoheadrightarrow \frac{R}{B_n} \simeq K[[x_1, \ldots, x_n]].$$

Restricted to $R'$ this homomorphism has image $K[x_1, \ldots, x_n]$. When restricted to $K[x_1, \ldots, x_m]$, for $m \geq n$, it coincides with the homomorphism defined by

$$K[x_1, \ldots, x_m] \twoheadrightarrow \frac{K[x_1, \ldots, x_m]}{(x_{n+1}, \ldots, x_m)} \simeq K[x_1, \ldots, x_n].$$

We will abuse notations and let $\rho_n$ denote both the function itself, and its restrictions to $R'$ and $K[x_1, \ldots, x_n]$.

The homomorphism $\rho_n$ is the "linear extension" of its restriction to the monoid $\mathcal{M}$ (which is not a $K$-vector space basis) in the sense that

$$\rho_n \left( \sum_{m \in \mathcal{M}} c_m m \right) = \sum_{m \in \mathcal{M}} c_m \rho_n(m) = \sum_{m \in \mathcal{M}^n} c_m m.$$

This is certainly not true for all homomorphisms from $R$; for instance, the quotient epimorphism $R \twoheadrightarrow R/(x_1, x_2, x_3, \ldots)$ vanishes at every monomial, but is not identically zero.

## 3.3 Initial ideals of generic ideals in $K[x_1, \ldots, x_n]$

In this section, we concern ourselves with the generic ideal

$$I = (f_1, \ldots, f_r) \subset K[x_1, \ldots, x_n]$$

generated by generic forms $f_i$ with total degree $d_i$. We note that the initial ideal is determined by the $d_i$; if $I' = (g_1, \ldots, g_r)$ is another generic ideal, generated by generic forms $g_i$ with $|g_i| = d_i$, then $\mathrm{gr}(I') = \mathrm{gr}(I)$. This holds for any admissible order, but, as stated above, we are interested in the case of the graded reverse lexicographic order.

To start, we establish two basic properties of the reverse lexicographic order:

**Lemma 3.3.1.** *If $h \in R'$ is homogeneous, and if $\nu$ is any positive integer, then either $\rho_\nu(h) = 0$, or $\mathrm{Lpp}(h) = \mathrm{Lpp}(\rho_\nu(h))$.*

In particular, the result holds for $h \in K[x_1, \ldots, x_n]$.

**Lemma 3.3.2.** *For any homogeneous ideal $J \subset R'$, and any positive integer $\nu$, we have that $\rho_\nu(\mathrm{gr}(J)) = \mathrm{gr}(\rho_\nu(J))$. The same formula holds for homogeneous ideals in $K[x_1, \ldots, x_n]$.*

*Proof.* It is enough to prove the assertion about ideals in $R'$. By [76, Lemma 3.3], we have that $\rho_\nu(\mathrm{gr}(J)) \subset \mathrm{gr}(\rho_\nu(J))$. It remains to prove the reverse inclusion. Let $0 \neq m \in \mathrm{gr}(\rho_\nu(J)) \cap \mathcal{M}^n$, then there exists a homogeneous element $h \in R'$ such that $m = \mathrm{Lpp}(\rho_\nu(h))$. By Lemma 3.3.1, $\mathrm{Lpp}(h) = m$. Clearly, $m \in \mathcal{M}^\nu$, so that $\rho_\nu(m) = m$. Therefore, $m \in \rho_\nu(\mathrm{gr}(J))$. □

We also need

**Lemma 3.3.3.** *The image of $I$ under the epimorphism $\rho_r$ is a generic ideal in $K[x_1, \ldots, x_r]$.*

In [55, Section I.3], Moreno defines the *stairs* (with respect to $>$) of I as

$$E(I) = \mathcal{M}^n \setminus (\mathrm{gr}(I) \cap \mathcal{M}^n)\,.$$

In passing, he notes:

**Proposition 3.3.4.** *If $r < n$, then the stairs are* cylindrical, *that is,* $E(I) = \tilde{E}^0 \times \mathbb{N}$ *where* $\tilde{E}^0 = E(I) \cap \mathcal{M}^{n-1}$.

**Corollary 3.3.5.** *If $r < n$, then the minimal generators of $\mathrm{gr}(I)$ are contained in $\mathcal{M}^{n-1}$.*

*Proof.* A monomial $m$ belongs to $E(I)$ iff the $\bar{m}$, the $x_1 \cdots x_{n-1}$ part of $m$, belongs to $E(I)$. Thus $m \not\in \mathrm{gr}(I)$ iff $\bar{m} \not\in \mathrm{gr}(I)$, so by contraposition we have that $m \in \mathrm{gr}(I)$ iff $\bar{m} \in \mathrm{gr}(I)$. Thus, if $m_1, \ldots, m_s$ is a set of monomial generators for $\mathrm{gr}(I)$, then so is $\bar{m}_1, \ldots, \bar{m}_s$. For a minimal generating set, we must have that $m_i = \bar{m}_i$ for $1 \leq i \leq s$, that is, $m_i \in \mathcal{M}^{n-1}$. □

In fact, we have

**Proposition 3.3.6.** *If* $r < n$*, then the minimal generators of* $\mathrm{gr}(I)$ *are contained in* $\mathcal{M}^r$*, and furthermore* $\mathrm{gr}(I) = \mathrm{gr}(\rho_r(I))^e$*, where the extension is to* $K[x_1, \ldots, x_n]$.

*Proof.* Clearly it is enough to show the last assertion. Since $I$ is a complete intersection, the Hilbert series of $K[x_1, \ldots, x_n]/I$ is given by $(1-t)^{-n} \prod_{i=1}^{r}(1-t^{d_i})$; this is also the Hilbert series of $K[x_1, \ldots, x_n]/\mathrm{gr}(I)$. Similarly, the Hilbert series of $K[x_1, \ldots, x_r]/\rho_r(I)$ is $(1-t)^{-r} \prod_{i=1}^{r}(1-t^{d_i})$, as is the Hilbert series of $K[x_1, \ldots, x_r]/\mathrm{gr}(\rho_r(I))$. By Lemma 3.3.2, restriction and the forming of initial ideals commute, so that $\mathrm{gr}(\rho_r(I))^e = \rho_r(\mathrm{gr}(I))^e$ may be regarded as a subideal of $\mathrm{gr}(I)$. Now,

$$\frac{K[x_1, \ldots, x_n]}{\mathrm{gr}(\rho_r(I))^e} \simeq \frac{K[x_1, \ldots, x_r]}{\mathrm{gr}(\rho_r(I))}[x_{r+1}, \ldots, x_n],$$

which has Hilbert series

$$\frac{1}{(1-t)^{(n-r)}} \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^r} = \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n}.$$

So $\mathrm{gr}(\rho_r(I))^e \subset \mathrm{gr}(I)$, and the two ideals have the same Hilbert series. Therefore, they must be equal. □

### 3.3.1 The complete structure of $\mathrm{gr}(I)$

Moreno discusses in [55, Conjecture I.4.1] a conjecture, which, if it holds true (and the computational evidence for its veracity is owerwhelming) completely determines the structure of the $\mathrm{gr}(I)$. The claim of the conjecture is as follows: by definition, $\mathrm{gr}(I)$ has minimal monomial generators $m_1, \ldots, m_\nu$. Denote by $\mathrm{gr}(I)_{<d}$ the monomial ideal generated by those $m_j$'s that have total degree $< d$. Then, the conjecture claims that the minimal monomial generators of degree $d$ are those monomials of $\mathcal{M}^n_d \setminus (\mathrm{gr}(I)_{<d} \cap \mathcal{M}^n_d)$ that occupy the (degrevlex) first $w(d)$ available spots, where $w(d)$ is determined by the difference of the Hilbert series of $\frac{K[x_1,\ldots,x_n]}{I}$ and the Hilbert series of $\frac{K[x_1,\ldots,x_n]}{\mathrm{gr}(I)_{<d}}$ in degree $d$. Note that the Hilbert series of $\frac{K[x_1,\ldots,x_n]}{I}$ is $\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n}$ and that the Hilbert series of the monomial algebra $\frac{K[x_1,\ldots,x_n]}{\mathrm{gr}(I)_{<d}}$ is easy to calculate (for instance using the ingenious method described

in [46]), that the series coincide in degrees $< d$, and that the latter series is no smaller than the generic series in degree $d$.

**Example 3.3.7.** Consider the generic ideal generated by two quadratic forms in 2 variables. The Hilbert series for the quotient is

$$\frac{(1-t^2)^2}{(1-t)^2} = 1 + 2t + t^2.$$

The zero ideal have Hilbert series

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \dots$$

These series differ by 2 in degree 2; therefore, the initial ideal should have two generators in degree 2. According to the conjecture, we should choose the 2 first (with respect to graded revlex), namely $x_1^2$ and $x_1x_2$. The Hilbert series for the monomial ideal generated by these two monomials is

$$\frac{1-2t^2+t^3}{(1-t)^2} = 1 + 2t + t^2 + t^3 + t^4 + t^5 + t^6 + \dots$$

which differ by 1 in degree 3 from the correct Hilbert series. We should thus add one cubic monomial. The first such monomial that is not divisible by $x_1^2$, nor by $x_1x_2$, is $x_2^3$. The Hilbert series for $\frac{K[x_1,x_2]}{(x_1^2,x_1x_2,x_2^3)}$ is $1 + 2t + t^2$, so we are done.

## *3.4 Initial ideals of generic ideals in* $R'$

We now generalize the results of the previous section to the ring $R'$. To that purpose, let $I = (f_1, \dots, f_r) \subset R'$ be a generic ideal, generated by generic forms $f_i$ with $\deg f_i = d_i$. As before, we note that the initial ideal is determined by the $d_i$'s.

From Proposition 3.3.6 we can determine the structure of (almost) all generic initial ideals $\mathrm{gr}(\rho_n(I))$:

**Proposition 3.4.1.** *For all non-negative integers* $s$*,*

$$\mathrm{gr}(\rho_r(I))^e = \mathrm{gr}(\rho_{r+s}(I))^e,$$

*where the extension is to* $R'$*.*

*Proof.* Let $r \le i < j$. Putting $n = j$ and applying Proposition 3.3.6 we get that

$$\mathrm{gr}(\rho_i(I))^e = \mathrm{gr}(\rho_j(I))$$

where the extension is to $K[x_1, \dots, x_j]$. The proof is finished by noting that we get the same result when we extend $\mathrm{gr}(\rho_i(I))$ from $K[x_1, \dots, x_i]$ to $K[x_1, \dots, x_j]$ to $R'$, or extend $\mathrm{gr}(\rho_i(I))$ to $R'$ directly. $\square$

We now use the theorem of degree-wise approximation from [76], which state that for all total degrees $d$, there exists an integer $N(d)$ such that, for any $n \geq N(d)$ we have that

$$\mathrm{gr}(I)_d = \mathrm{gr}(\rho_n(I))_d^e$$

where the right-hand side is extended to $R'$ using the natural inclusion. Since $\mathrm{gr}(\rho_n(I))_d^e$ stabilizes for $n \geq r$, for any $d$, we conclude:

**Theorem 3.4.2.** *For $n \geq r$, $\mathrm{gr}(\rho_n(I))^e = \mathrm{gr}(I)$. Thus, $\mathrm{gr}(I)$ is generated in $\mathcal{M}^r$, and is finitely generated.*

We can avoid the use of the approximation theorem, by arguing as follows: since

$$\rho_n(\mathrm{gr}(I)) = \mathrm{gr}(\rho_n(I)) = \mathrm{gr}(\rho_r(I))^e,$$

where the last extension is to $K[x_1, \ldots, x_n]$, we must have that any monomial $m \in \rho_n(\mathrm{gr}(I))$ is divisible by a monomial $t \in \mathrm{gr}(\rho_r(I))$. Every monomial in $\mathrm{gr}(I)$ lies in some $\rho_n(\mathrm{gr}(I))$; hence, the result follows.

## 3.5 Initial ideals of "almost" generic ideals in $K[x_1, \ldots, x_n]$

### 3.5.1 The associated homogeneous ideal

For any $f$ in the ring $K[x_1, \ldots, x_n]$ we denote by $c(f)$ the homogeneous component of $f$ of maximal degree. If $I \subset K[x_1, \ldots, x_n]$ is an ideal, we denote by $c(I)$ the homogeneous ideal generated by all $c(f)$ for $f \in I$. This homogeneous ideal is the graded associated ideal with respect to the total degree filtration; since the initial ideal $\mathrm{gr}(I)$ is the graded associated ideal to the filtration induced by Lpp, and since this latter filtration is a refinement of the total-degree filtration, we have that $\mathrm{gr}(I) = \mathrm{gr}(c(I))$. We can also see this directly: for any $f \in I$, we have that $\mathrm{Lpp}(f) = \mathrm{Lpp}(c(f))$.

It is well known that not every generating set of an ideal is a Gröbner Basis. Similarly, not every generating set $F$ of $I$ has the property that $\{c(f) | f \in F\}$ generates $c(I)$. In the generic case, however, we have the following:

**Lemma 3.5.1.** *Let $J = (f_1, \ldots, f_r) \subset K[x_1, \ldots, x_n]$, with $r \leq n$. Suppose that all $c(f_i)$'s are generic, as is $I = (c(f_1), \ldots, c(f_r))$. Then $c(J) = I$.*

*Proof.* Assume, towards a contradiction, that there exists an $f \in J$ such that $c(f) \in c(J) \setminus I$. Let $d = |f|$. Since $f \in J$ we can write $f = \sum_{i=1}^{r} q_i f_i$. Furthermore, since $c(f) \notin I$, we must have that $\max_i |q_i f_i| > d$. Put

$$\mathcal{S} = \left\{ S = (a_1, \ldots, a_r) \middle| a_i \in K[x_1, \ldots, x_n],\ f = \sum_{i=1}^{r} a_i f_i,\ \max_i |a_i f_i| > d \right\}.$$

For $S \in \mathcal{S}$, put $\delta_S = \max_i |a_i f_i|$. By assumptions, $\delta_S > d$, and $\mathcal{S}$ is non-empty, containing the element $(q_1,\dots,q_r)$. Since the set $\{\delta_S | S \in \mathcal{S}\}$ is a non-void subset of the natural numbers, it contains a minimum. Choose an $S = (a_1,\dots,a_r)$ where that minimum is obtained.

Now, the $c(f_i)$'s form a regular sequence, so all syzygies involving them are trivial (see [53, Theorem 16.5]) . We apply this to the homogeneous component of maximal $\delta_S$-degree in

$$f = \sum_{i=1}^{r} a_i f_i.$$

Denoting by $V \subset \{1,\dots,r\}$ the set of the indices for which $|a_i f_i| = \delta_S$, we get that

$$0 = \sum_{v\in V} c(a_v)c(f_v). \tag{3.1}$$

For simplicity of notations, we assume that $V = \{1,\dots,s\}$ for $s \le r$. From (3.1) we see that $(c(a_1),\dots,c(a_s))$ is a syzygy to $(c(f_1),\dots,c(f_s))$. It must be a trivial one, that is, it must be a linear combination of vectors

$$(0,\dots,c(f_w),0,\dots,-c(f_v),0,\dots)$$

with non-zero entries in positions $v$ and $w$. Summing up, we have that

$$\begin{aligned}(c(a_1),\dots,c(a_s)) = {} & \mu_{12}\times(c(f_2),-c(f_1),0,\dots,0)\\ & +\mu_{13}\times(c(f_3),0,-c(f_1),0,\dots,0)\\ & \qquad\vdots\\ & +\mu_{s-1,s}\times(0,\dots,c(f_s),-c(f_{s-1}))\end{aligned}$$

We conclude that

$$\forall v \in V: \quad c(a_v) = \sum_{w\in V} e_{vw}c(f_w),$$

where the homogeneous $e_{vw}$'s fulfill $e_{vw} = -e_{wv}$, $e_{vv} = 0$. By defining $e_{ij} = 0$ whenever $(i,j) \notin V \times V$, we get an $r \times r$ skew-symmetric matrix $E = (e_{ij})$ such that

$$\forall 1 \le i \le r: \quad c(a_i) = \sum_{j=1}^{r} e_{ij}c(f_j).$$

Since $E$ is skew-symmetric, for all vectors $x = (x_1,\dots,x_r)$ in the r-fold cartesian product $K[x_1,\dots,x_n]^r$ we have that $xEx^t = 0$. We apply this to the vector $(f_1,\dots,f_r)$, and get that

$$\sum_{i=1}^{r}\sum_{j=1}^{r} e_{ij}f_if_j = 0.$$

The conclusion draws near. Using the above, we write

$$f = \sum_{i=1}^{r} a_i f_i = \sum_{i=1}^{r} a_i f_i - \sum_{i=1}^{r}\sum_{j=1}^{r} e_{ij} f_i f_j = \sum_{i=1}^{r}\left(a_i - \sum_{j=1}^{r} e_{ij} f_j\right) f_i.$$

Now put

$$b_i = a_i - \sum_{j=1}^{r} e_{ij} f_j, \qquad 1 \le i \le r.$$

Since

$$c(a_i) = \sum_{j=1}^{r} e_{ij} c(f_j)$$

we get that $|b_i| < |a_i|$ hence that $|b_i f_i| < \delta_S$. But then $f = \sum_{i=1}^{r} b_i f_i$ and $T = (b_1, \dots, b_r) \in \mathcal{S}$ with a $\delta_T < \delta_S$. This contradicts the minimality of $\delta_S$. □

*Remark 3.5.2.* It follows from our discussion above that

$$\mathrm{gr}(J) = \mathrm{gr}(c(I)) = \mathrm{gr}(I).$$

*Remark 3.5.3.* The fact that the syzygies of $(c(f_1), \dots, c(f_r))$, and of any of its sub-vectors, are trivial, follows from the fact that the syzygies may be viewed as the kernel $\mathcal{K}$ in the following exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{i=1}^{r} K[x_1, \dots, x_n] T_i^{(1)} \longrightarrow (c(f_1), \dots, c(f_r)) \longrightarrow 0 \qquad (3.2)$$

where the non-trivial map is defined by $T_i^{(1)} \mapsto c(f_i)$ and extended linearly. (3.2) is the start of the Koszul complex on the elements $c(f_1), \dots, c(f_r)$, the next step being

$$\bigoplus_{1 \le a < b \le r} K[x_1, \dots, x_n] T_{ij}^{(2)} \longrightarrow \bigoplus_{i=1}^{r} K[x_1, \dots, x_n] T_i^{(1)}.$$

The relevant boundaries are generated by $c(f_a) T_b^{(1)} - c(f_b) T_a^{(1)}$, so the phrase "all syzygies are trivial" means precisely that the first homology group of the Koszul complex vanishes. That this is so for complete intersections is showed in [53, Theorem 16.5].

### 3.5.2 "Almost generic" ideals

What happens if we start with a generic ideal, generated by generic forms, and then replace some of the coefficients of the monomials in the forms with non-generic values?

In Lemma 3.5.4, we study what happens when we leave the coefficients of monomials in $\mathcal{M}^r$ as they are, but manipulate the others:

**Lemma 3.5.4.** *Let* $I$ *be a generic ideal in* $K[x_1, \dots, x_n]$ *generated by generic forms* $f_1$ *to* $f_r$*, with* $r < n$*. For* $1 \leq i \leq r$, $d_i = |f_i|$*, let* $g_i \in K[x_1, \dots, x_n]$ *be homogeneous of degree* $d_i$*, and such that each monomial in* $\mathrm{Mon}(g_i)$ *is divisible by at least one of the variables* $x_{r+1}, \dots, x_n$*; put* $h_i = f_i + g_i$*. Denote by* $J$ *the ideal* $(h_1, \dots, h_r) \subset K[x_1, \dots, x_n]$*. Then* $\mathrm{gr}(I) = \mathrm{gr}(J)$*.*

*Proof.* We know from previous results that $\mathrm{gr}(I) = \mathrm{gr}(\rho_r(I)^e)$, where the extension is to $K[x_1, \dots, x_n]$. Since it will simplify our proof, we henceforth assume that the $f_i$'s are generic forms in $K[x_1, \dots, x_r]$. Let $m \in \mathrm{gr}(I) \cap \mathcal{M}^n$, $m \neq 0$. Then there exists a $g \in I$ with $\mathrm{Lpp}(g) = m$. We can write $g = \sum_{i=1}^r e_i f_i$, where the $e_i$'s are homogeneous. Put

$$h = \sum_{i=1}^{r} e_i h_i = \sum_{i=1}^{r} e_i f_i + \sum_{i=1}^{r} e_i g_i.$$

Clearly, each monomial in $\mathrm{Mon}(\sum_{i=1}^r e_i f_i)$ is greater than any monomial in $\mathrm{Mon}(\sum_{i=1}^r e_i g_i)$. It follows that

$$\mathrm{Lpp}(h) = \mathrm{Lpp}(g) = m.$$

Therefore, $m \in \mathrm{gr}(J)$.

We have showed that $\mathrm{gr}(I) \subset \mathrm{gr}(J)$. Since $I$ is generic, the quotient $\frac{K[x_1, \dots, x_n]}{I}$ has (lexicographically) minimal Hilbert series among all quotients of $K[x_1, \dots, x_n]$ by a homogeneous ideal generated by forms of degree $d_1$ to $d_r$. This useful property was shown by Fröberg in [27], and is to be interpreted in the following way: if we write the Hilbert series of the generic quotient as $\sum_{k=0}^\infty \nu_k t^k$ and the Hilbert series of the other algebra as $\sum_{k=0}^\infty w_k t^k$, then if the set $\{\nu_k - w_k | k \in \mathbb{N}\} \setminus \{0\}$ is non-empty, and if $k$ is the smallest $k$ such that $\nu_k \neq w_k$, then $\nu_k < w_k$.

The ideal $J$ belongs to the prescribed class of homogeneous ideals. Therefore, $\frac{K[x_1, \dots, x_n]}{J}$ have a Hilbert series that is no smaller than the Hilbert series of $\frac{K[x_1, \dots, x_n]}{I}$, hence $\frac{K[x_1, \dots, x_n]}{\mathrm{gr}(J)}$ have a Hilbert series that is no smaller than that of $\frac{K[x_1, \dots, x_n]}{\mathrm{gr}(I)}$. This shows that the inclusion $\mathrm{gr}(I) \subset \mathrm{gr}(J)$ can not be strict. $\square$

If on the other hand $J$ is obtained from $I$ by destroying the genericity of the *highest* variables, then we can not hope to get the same initial ideal. We believe, however, that the initial ideal is generated in the $r + \nu$ first variables, where $\nu$ denotes the index of the last variable that is manipulated:

**Conjecture 3.5.5.** *Let* $J \subset K[x_1, \dots, x_\nu, x_{\nu+1}, \dots, x_{\nu+s}]$ *be a homogeneous ideal generated by homogeneous* $f_1, \dots, f_r$ *with* $r \le s$. *Assume that the polynomials* $\bar{f}_1, \dots, \bar{f}_r$ *generate a generic ideal in*

$$K[x_{\nu+1}, \dots, x_{\nu+s}] \simeq \frac{K[x_1, \dots, x_\nu, x_{\nu+1}, \dots, x_{\nu+s}]}{(x_1, \dots, x_\nu)},$$

*where* $\bar{f}_i$ *denotes the image of* $f_i$. *Then the monoid ideal* $\mathrm{gr}(J) \cap \mathcal{M}^{\nu+s}$ *is minimally generated in* $\mathcal{M}^{\nu+r}$.

*Remark 3.5.6.* The conjecture is easily seen to be true in the two "extreme cases": when the $f_i$'s are generic forms in $K[x_1, \dots, x_\nu, x_{\nu+1}, \dots, x_{\nu+s}]$, we have that $\mathrm{gr}(J) \cap \mathcal{M}^{\nu+s}$ is generated in $\mathcal{M}^r$; when $f_i = \bar{f}_i$ for all $i$, clearly $\mathrm{gr}(J) \cap \mathcal{M}^{\nu+s}$ is generated in the commutative monoid on the letters $x_{\nu+1}, \dots, x_{\nu+r}$ and in particular in $\mathcal{M}^{\nu+r}$. The author has checked several other examples by computer.

### 3.5.3 Initial ideal generic ideals with "ordered coefficients"

Let $n$, $r$ and $d_1, \dots, d_r$ be positive integers, and define $t_n = \sum_{i=1}^{r} \binom{n+d_i-1}{n-1}$. Then $t_n$ is the cardinality of the disjoint union of the set of monomials of degree $d_i$ in $K[x_1, \dots, x_n]$, for $i \le i \le r$. We can therefore define

$$f_i := \sum_{m \in \mathcal{M}^n_{d_i}} y_{i,m} m \in S_n,$$

where

$$S_n = K[\{y_{i,m}\}][x_1, \dots, x_n] \simeq K[y_1, \dots, y_{t_n}][x_1, \dots, x_n],$$

and put $I = (f_1, \dots, f_r)$. The ordering of the $y_i$'s is such that $y_1, \dots, y_{t_1}$ are the variables that occur together with monomials in $\mathcal{M}^1$, $y_{t_1+1}, \dots, y_{t_2}$ together with monomials in $\mathcal{M}^2 \setminus \mathcal{M}^1$, and so forth. We say that the $f_i$'s are *pure generic forms*, and that $I$ is a *pure generic ideal*.

**Example 3.5.7.** If $r = n = d_1 = d_2 = 2$ then $f_1 = y_1 x_1^2 + y_3 x_1 x_2 + y_4 x_2^2$, and $f_2 = y_2 x_1^2 + y_5 x_1 x_2 + y_6 x_2^2$.

Let $>$ be the total degree, then reverse lexicographic order on $S_n$, when the $Y$-variables are given weight $0$. This is the same as taking the "twisted" product order of revlex on the two submonoids on the $y$'s and on the $x$'s. That is, when comparing two monomials $tm$ and $t'm'$, with $t, t' \in [y_1, \dots, y_{t_n}]$, $m, m' \in [x_1, \dots, x_n]$, we first compare $m$ and $m'$, and only if they are equal do we compare $t$ and $t'$. Here, $[x_1, \dots, x_n]$ denotes the free commutative monoid on the letters $x_1, \dots, x_n$, and similarly for the $y_j$'s.

The following lemma is obvious:

**Lemma 3.5.8.** *Let $>_{\text{rev}}$ be the ordinary degrevlex order on $S_n$ (that is, when the $y$-variables are given weight 1), let $f \in S_n$ be bi-homogeneous with respect the two groups of variables, and let $J \subset S_n$ be generated by such bi-homogeneous elements. Then* $\text{Lpp}_>(f) = \text{Lpp}_{>_{\text{rev}}}(f)$, *and* $\text{gr}_>(J) = \text{gr}_{>_{\text{rev}}}(J)$.

In particular, this holds for the pure generic ideal $I$.

For any $1 \leq \nu < n$, we denote by $\rho_{*,\nu}$ the epimorphism

$$S_n \twoheadrightarrow S_n/(x_{\nu+1}, \dots, x_n) \simeq K[y_1, \dots y_{t_n}][x_1, \dots, x_\nu]. \tag{3.3}$$

We need "bi-graded" counterparts of Lemma 3.3.1 and Lemma 3.3.2. Since these results hold in a more general setting, namely in the ring $K[Y][[X]]'$, defined in Section 3.6, we do not give proofs here, but refer to the proofs of the more general Lemma 3.6.4 and Lemma 3.6.5.

The ring $S_n$ can be regarded as a polynomial ring on two groups of variables, and having coefficients in $K$, that is, as

$$S_n = K[y_1, \dots, y_{t_n}; x_1, \dots, x_n].$$

It can also be regarded as a polynomial ring on the variables $x_1, \dots, x_n$, with coefficients in the domain $K[y_1, \dots, y_{t_n}]$. If an element $f \in S_n$ is homogeneous when $S_n$ is viewed in this later fashion, we say that $f$ is *$\mathcal{M}$-homogeneous*. We will make use of this notion also in the ring $K[Y][[X]]'$. There, we will also speak about *$\mathcal{M}$-locally finitely generated ideals*. The meaning is the same: we give the Y-variables weight zero, and then check for homogeneity or locally finiteness.

**Lemma 3.5.9.** *If $h \in S_n$ is $\mathcal{M}$-homogeneous, and if $1 \leq \nu \leq n$, then either $\rho_{*,\nu}(h) = 0$, or* $\text{Lpp}(h) = \text{Lpp}(\rho_{*,\nu}(h))$.

**Lemma 3.5.10.** *For any $\mathcal{M}$-homogeneous ideal $J \subset S_n$, and for $1 \leq \nu \leq n$, we have that* $\rho_{*,\nu}(\text{gr}(J)) = \text{gr}(\rho_{*,\nu}(J))$.

The following lemma is a key ingredient in the proof of the generalization of Proposition 3.3.6:

**Lemma 3.5.11.** *If $r \leq n$ then $S_n/I$ is a complete intersection.*

*Proof.* Let $V \subset \{y_1, \dots, y_{t_n}\}$ be the set of all variables $y_\nu$ except those that occur as the coefficient of $x_i^{d_i}$ in $f_i$, and let $J$ be the ideal generated by $V$. If we re-order the Y-variables so that $V = \{y_{r+1}, \dots, y_{t_n}\}$ and $\{y_1, \dots, y_r\}$ are those Y-variables not in V, then the image $\bar{f}_i$ of $f_i$ in $S_n/J$ is $y_i x_i^{d_i}$. Therefore,

$$\frac{S_n}{I+J} \simeq \frac{K[y_1, \dots, y_r; x_1, \dots, x_n]}{(y_1 x_1^{d_1}, \dots, y_r x_r^{d_r})},$$

which is a complete intersection because the support of the monomials are disjoint; so it has Hilbert series $\frac{\prod_{i=1}^r (1-z^{(1+d_i)})}{(1-z)^{n+r}}$. We have that

$$I + J = (f_1, \dots, f_r, y_{r+1}, \dots, y_{t_n}).$$

Since $S_n/(I+J)$ has Hilbert series

$$\frac{\prod_{i=1}^r (1 - z^{(1+d_i)})}{(1-z)^{(n+r)}} = \frac{(1-z)^{(t_n - r)} \prod_{i=1}^r (1 - z^{(1+d_i)})}{(1-z)^{(n+t_n)}},$$

it follows that $f_1, \dots, f_r, y_{r+1}, \dots, y_{t_n}$ must be a regular sequence in $S_n$. Therefore, $f_1, \dots, f_r$ is also a regular sequence, hence $S_n/I$ is a complete intersection. □

**Proposition 3.5.12.** *If* $r \leq n$*, then the minimal monomial generators of* $\mathrm{gr}(I)$ *are contained in* $[y_1, \dots, y_{t_n}] \oplus \mathcal{M}^r$*, and furthermore* $\mathrm{gr}(I) = \mathrm{gr}(\rho_{*,r}(I))^e$*, where the extension is to* $S_n$*.*

*Proof.* By Lemma 3.5.11, $S_n/I$ is a complete intersection; it follows from this that so is

$$K[y_1, \dots, y_{t_n}][x_1, \dots, x_r]/\rho_{*,r}(I).$$

Therefore, their bi-graded Hilbert series are, respectively,

$$(1-u)^{-t_n}(1-v)^{-n} \prod_{i=1}^r (1 - uv^{d_i})$$

and

$$(1-u)^{-t_n}(1-v)^{-r} \prod_{i=1}^r (1 - uv^{d_i});$$

this is also the bi-graded Hilbert series of $S_n/\mathrm{gr}(I)$ and

$$K[y_1, \dots, y_{t_n}][x_1, \dots, x_r]/\mathrm{gr}(\rho_{*,r}(I)).$$

Furthermore, we have that

$$\frac{K[y_1, \ldots, y_{t_n}][x_1, \ldots, x_n]}{\mathrm{gr}(\rho_{*,\nu}(I))^e} \simeq \frac{K[y_1, \ldots, y_{t_n}][x_1, \ldots, x_r]}{\mathrm{gr}(\rho_{*,\nu}(I))}[x_{r+1}, \ldots, x_n],$$

hence this quotient has bi-graded Hilbert series

$$(1-\nu)^{-(n-r)}(1-u)^{-t_n}(1-\nu)^{-r}\prod_{i=1}^{r}(1-u\nu^{d_i}) =$$
$$= (1-u)^{-t_n}(1-\nu)^{-n}\prod_{i=1}^{r}(1-u\nu^{d_i}).$$

By Lemma 3.5.10, we can regard $\mathrm{gr}(\rho_{*,r}(I))^e$ as a subideal of $\mathrm{gr}(I)$ (the extension is to $S_n$). Since these ideals have the same bi-graded Hilbert series, they are equal. $\square$

Since $\rho_{*,r}(I)$ is generated in $S_r$, we must have that a minimal Gröbner basis of the ideal is contained in that subring of $S_n$. Therefore:

**Corollary 3.5.13.** *If $r < n$, then the minimal monomial generators of $\mathrm{gr}(I)$ are contained in $[y_1, \ldots, y_{t_r}] \oplus \mathcal{M}^r$.*

### 3.5.4 Examples

These examples were calculated using the computer algebra programs Macaulay 2 and Maple ([38, 24]). We used the field $GF(31991)$ as coefficient field, but the results should hold whenever $\mathrm{char}(K) \neq 2, 3$. We assume that $n \geq r = 2$.

**Example 3.5.14.** Let $J$ be a pure generic ideal,

$$J \subset K[\{\alpha_{ij}\} \cup \{\beta_{ab}\}][x_1, \ldots, x_n],$$

and suppose that $J$ is generated by the two pure generic quadratic forms

$$f = \sum_{i=1}^{n}\sum_{j=i}^{n} \alpha_{ij}x_i x_j$$
$$g = \sum_{a=1}^{n}\sum_{b=a}^{n} \beta_{ab}x_a x_b$$

Then

$$\mathrm{gr}(J) = (x_1^2\beta_{11},\ x_1^2\alpha_{11},\ x_1x_2\alpha_{12}\beta_{11},\ x_1x_2^2\alpha_{22}\beta_{11},\ x_1x_2^2\alpha_{11}\alpha_{22}\beta_{12},\ x_2^3\alpha_{11}^2\beta_{22}^2).$$

**Example 3.5.15.** Let

$$J \subset K[\{\alpha_{ij}\} \cup \{\beta_{abc}\}][x_1, \dots, x_n]$$

be the pure generic ideal generated by a pure generic quadratic form

$$f = \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} x_i x_j$$

and a pure generic cubic form

$$g = \sum_{a=1}^{n} \sum_{b=a}^{n} \sum_{c=b}^{n} \beta_{abc} x_a x_b x_c.$$

Then

$$\begin{aligned} \mathrm{gr}(J) = (&x_1^2\alpha_{11},\ x_1^3\beta_{111},\ x_1^2x_2\alpha_{12}\beta_{111},\ x_1x_2^2\alpha_{11}^2\beta_{122},\ x_1^2x_2^2\alpha_{22}\beta_{111}, \\ &x_1x_2^3\alpha_{11}\alpha_{12}\beta_{222},\ x_1x_2^3\alpha_{11}^2\beta_{222},\ x_1x_2^3\alpha_{12}^2\beta_{111}\beta_{222},\ x_2^4\alpha_{11}^3\beta_{222}^2). \end{aligned}$$

## 3.6 Initial ideals of generic ideals in $K[Y][[X]]'$

If X is any set, and C is a commutative ring, then we denote by [X] the free commutative monoid on X.We denote by C[[X]] the power series ring on X with coefficients in C, that is, the set of all maps $[X] \to C$, with point-wise addition and multiplication with scalars, and with multiplication given by the natural *convolution*. We denote by C[X] the polynomial ring on X with coefficients in C, that is, the subset of C[[X]] consisting of all *finitely supported* maps.

There is a homomorphism of abelian monoids $|\cdot| : [X] \to (\mathbb{N}, +)$ which is uniquely determined by the demand that $|X| = \{1\}$. We call the value of this homomorphism on a monomial $m \in [X]$ the *total degree* of the monomial. We denote by $C[[X]]'$ the subset of C[[X]] consisting of all elements for which there exists a common bound for the total degree of the monomial occuring in the support.

We now let $X = \{x_1, x_2, x_3, \dots\}$ and $Y = \{y_1, y_2, y_3, \dots\}$. Then $\mathcal{M} = [X]$ and $R' = K[[X]]'$. To generalize the results of Section 3.5.3, we consider the ring $S := K[Y][[X]]'$. The underlying monoid is $[Y] \oplus [X]$, which we order by the (graded) reverse lexicographic order where the Y-variables are given weight 0, that is, by the "twisted" product order of revlex on the two subsemigroups. There is no problem in finding leading power products in this ring.

For any $f \in K[Y][[X]]'$, we denote by $\mathrm{Mon}(f) \subset [Y] \oplus [X]$ the set of XY-monomials occuring with non-zero coefficient in f.

*Remark 3.6.1.* Note that

$$R'[Y] = K[[X]]'[Y] \subsetneq K[Y][[X]]',$$

since

$$\sum_{i=1}^{\infty} y_i x_i \in K[Y][[X]]' \setminus R'[Y].$$

Let $r, d_1, \ldots, d_r, n$ be positive integers, let $t_n$ be as above, that is, $t_n = \sum_{i=1}^{r} \binom{n+d_i-1}{n-1}$. Set

$$f_{i,n} := \sum_{m \in \mathcal{M}^n_{d_i}} y_{i,m} m \in K[\{y_{i,m}\}][x_1, \ldots, x_n] \simeq$$

$$\simeq K[y_1, \ldots, y_{t_n}][x_1, \ldots, x_n] = S_n \quad (3.4)$$

Then there exists $f_1, \ldots, f_r \in K[Y][[X]]'$ such that for each $i, v$, $\rho_{*,v}(f_i) = f_{i,v}$. We have here generalized the definition of $\rho_{*,v}$ given in (3.3), so that $\rho_{*,v}$ is the quotient epimorphism

$$S \twoheadrightarrow S/C_v \simeq K[Y][x_1, \ldots, x_v], \quad (3.5)$$

where $C_v$ is the ideal generated by all power series in $K[[x_{v+1}, x_{v+2}, x_{v+3}, \ldots]]$ with zero constant term. This coincides with the former definition on $S_n$. Note that the ordering of the Y-variables is defined so that

$$S_n \subset S_{n+1} \subset S_{n+2} \subset \cdots \subset S$$

for all $n$, and that the $S_n$'s form an increasing, exhaustive filtration on S.

**Example 3.6.2.** Let $r = d_1 = d_2 = 2$. Then

$$f_1 = y_1 x_1^2 + y_3 x_1 x_2 + y_4 x_2^2 + y_7 x_1 x_3 + y_8 x_2 x_3 + y_9 x_3^2 + \ldots$$
$$f_2 = y_2 x_1^2 + y_5 x_1 x_2 + y_6 x_2^2 + y_{10} x_1 x_3 + y_{11} x_2 x_3 + y_{12} x_3^2 + \ldots$$

Now put $I = (f_1, \ldots, f_r)$. We say that I is a *pure generic ideal* in $K[Y][[X]]'$. If $n > r$, then $\rho_{*,n}(I)$ is an ideal in $K[Y][x_1, \ldots, x_n]$ but generated in $S_n$, so that, by Proposition 3.5.12, we have that $\mathrm{gr}(\rho_{*,n}(I))$ is generated in $S_r$, and is in fact the extension of $\mathrm{gr}(\rho_{*,r}(I))$. Note also that Lemma 3.5.8 holds also in this more general situation, and that I is bi-homogeneous. Therefore, we have that

**Lemma 3.6.3.** *The initial ideal* $\mathrm{gr}(I)$ *w.r.t* $>$ *coincides with the initial ideal with respect to the graded revlex order on* $K[Y][[X]]'$ *(when the* $Y$*-variables have weight 1).*

The following two lemmas generalize Lemma 3.5.9 and Lemma 3.5.10:

**Lemma 3.6.4.** *If* $h \in K[Y][[X]]'$ *is* $\mathcal{M}$*-homogeneous (that is, homogeneous when the* $Y$*-variables are given weight zero), and if* $\nu$ *is any positive integer, then either* $\rho_{*,\nu}(h) = 0$*, or* $\mathrm{Lpp}(h) = \mathrm{Lpp}(\rho_{*,\nu}(h))$.

*Proof.* If $\rho_{*,\nu}(h) \neq 0$ then $[Y] \oplus \mathcal{M}^\nu \supset \mathrm{Mon}(\rho_{*,\nu}(h)) \neq \emptyset$. Using the properties of the order $>$, we see that if $m \in [Y] \oplus \mathcal{M}^\nu$ and

$$m' \in ([Y] \oplus \mathcal{M}) \setminus ([Y] \oplus \mathcal{M}^\nu)$$

have the same total degree with respect to $\mathcal{M}$, then $m > m'$. Therefore, $\mathrm{Lpp}(\rho_{*,\nu}(h))$, which is greater than any other monomial in $\mathrm{Mon}(\rho_{*,\nu}(h))$, is also greater than the remaining monomials in

$$\mathrm{Mon}(h) \cap \big(([Y] \oplus \mathcal{M}) \setminus ([Y] \oplus \mathcal{M}^\nu)\big).$$

□

**Lemma 3.6.5.** *For any* $\mathcal{M}$*-homogeneous ideal* $J \subset K[Y][[X]]'$*, and any positive integer* $\nu$*, we have that* $\rho_{*,\nu}(\mathrm{gr}(J)) = \mathrm{gr}(\rho_{*,\nu}(J))$.

*Proof.* If $f \in J$, $m = \mathrm{Lpp}(f)$, then either $\rho_{*,\nu}(m) = 0$, or $\rho_{*,\nu}(m) = m$. In the latter case, $m = \mathrm{Lpp}(\rho_{*,\nu}(f))$. We have shown that $\rho_{*,\nu}(\mathrm{gr}(J)) \subset \mathrm{gr}(\rho_{*,\nu}(J))$.

Conversely, let $m \in \mathrm{gr}(\rho_{*,\nu}(J))$, $m = \mathrm{Lpp}(\rho_{*,\nu}(h))$, $h \in J$. Suppose that $m \neq 0$. Then $\rho_{*,\nu}(h) \neq 0$, so by Lemma 3.6.4 we conclude that $\mathrm{Lpp}(h) = \mathrm{Lpp}(\rho_{*,\nu}(h)) = m$. Therefore, $m \in \mathrm{gr}(J)$. Clearly $m = \rho_{*,\nu}(m)$, thus $m \in \rho_{*,\nu}(\mathrm{gr}(J))$. □

**Theorem 3.6.6.** *The minimal monomial generators of* $\mathrm{gr}(I)$ *are contained in* $[y_1, \ldots, y_{t_r}] \oplus \mathcal{M}^r$*, and furthermore* $\mathrm{gr}(I) = \mathrm{gr}(\rho_{*,r}(I))^e$*, where the extension is to* $S$.

*The* coefficient ideals $(\mathrm{gr}(I) : m) \cap K[Y]$*,* $m \in \mathcal{M}$*, are finitely generated monomial ideals generated in* $K[y_1, \ldots, y_{t_r}]$*. There are but finitely many different such ideals.*

*Proof.* As observed above, for any $n > r$ we have that

$$\mathrm{gr}(\rho_{*,n}(I)) = \mathrm{gr}(\rho_{*,r}(I))^e,$$

where the extension is to $S_n$. Since, by Lemma 3.6.5,

$$\mathrm{gr}(\rho_{*,n}(I)) = \rho_{*,n}(\mathrm{gr}(I)),$$

and since for any monomial

$$m \in \operatorname{gr}(I) \cap ([Y] \oplus [X])$$

there is an $N$ such that

$$n > N \implies m \in \rho_{*,n}(\operatorname{gr}(I)),$$

we get that $\operatorname{gr}(I)$ is generated in $K[Y][x_1, \ldots, x_r]$, and that

$$\operatorname{gr}(I) = \operatorname{gr}(\rho_{*,r}(I))^e.$$

It follows that $\operatorname{gr}(I)$ is generated in $K[y_1, \ldots, y_{t_r}][x_1, \ldots, x_r]$.

We then have that

$$\operatorname{gr}(I) = (t_1 m_1, \ldots, t_s m_s),\ t_i \in [y_1, \ldots, y_{t_r}],\ m_i \in [x_1, \ldots, m_r].$$

For any $m \in \mathcal{M}$, we have that $(\operatorname{gr}(I) : m) \cap K[Y]$ is generated by those $t_i$ for which $m_i | m$. Thus, the coefficient ideal depends only indirectly on $m$: it is determined by the set $\{ i \in \{1, \ldots, r\} | m_i | m \}$. There are only finitely many subset of $\{1, \ldots, r\}$, hence the last assertion. □

**Example 3.6.7.** Let $J$ be the pure generic ideal of Example 3.5.14 (but in the ring $K[Y][[X]]'$, so that $n = \infty$). Then the different coefficient ideals are

$$(\alpha_{11}, \beta_{11}),\ (\alpha_{11}\beta_{12}, \alpha_{11}\beta_{22}, \alpha_{12}\beta_{11}\beta_{22}),\ (\alpha_{11}\beta_{12}),\ (\alpha_{11}^2\beta_{22}^2),\ \text{and } 0.$$

### *3.6.1 Regarding the $Y$-variables as coefficients in a domain*

The admissible order $>$ on $[Y] \oplus [X]$ restricts to the ordinary reverse lexicographic order $>_x$ on $[X]$. We may regard $K[Y][[X]]'$ as having underlying monoid $[X]$, ordered by $>_x$, and having coefficients in the domain $K[Y]$. We define the *initial term* of an element $f \in K[Y][[X]]'$ as

$$\operatorname*{in}_{>_x}(f) = \operatorname*{lc}_{>_x}(f) \operatorname*{Lpp}_{>_x}(f),$$

where the leading coefficient $\operatorname{lc}_{>_x}(f) \in K[Y]$, and the leading power product $\operatorname{Lpp}_{>_x}(f) \in [X]$. We define the initial ideal with respect to $>_x$ of an ideal $J \subset K[Y][[X]]'$ as

$$\operatorname*{in}_{>_x}(J) = \left\{ \operatorname*{in}_{>_x}(f) \middle| f \in J \right\}.$$

Since $K[Y]$ is a domain, but not a field, the so called *coefficient ideals*

$$(\operatorname*{in}_{>_x}(J) : m) \cap K[Y], \quad m \in [X],$$

may be different from $0$ and $K[Y]$; in fact, they can even be non-finitely generated.

Similarly, if $E \subset K[Y][[X]]'$ is any set, then $\mathrm{gr}(E)$ and $\mathrm{in}_{>_x}(E)$ denotes the set of leading power products and the initial terms of $E$.

We will occasionally study the restriction of $>$ to $[Y]$; this term order we denote by $>_y$, and leading power products of elements $g \in K[Y]$ we denote by $\mathrm{Lpp}_{>_y}(g) \subset [Y]$.

Localizing $K[Y]$ in the multiplicatively closed set $K[Y] \setminus \{0\}$, we get its field of fractions $K(Y)$. There is a canonical inclusion

$$K[Y][[X]]' \hookrightarrow K(Y)[[X]]'.$$

The ring $K(Y)[[X]]$ is similar to $R'$; we have simply replaced $K$ with an overfield $K(Y)$. Therefore, leading power products and initial ideals in $K(Y)[[X]]'$ are defined in the usual fashion.

*Remark 3.6.8.* We do *not*, as in the polynomial ring case, get the ring $K(Y)[[X]]'$ by localizing the ring $K[Y][[X]]'$ in the multiplicatively closed set $K[Y] \setminus \{0\}$, since the resulting ring does not contain i.e. the element $\sum_{j=1}^{\infty} x_j/y_j$.

**Lemma 3.6.9.** *Let* $J \subset K[Y][[X]]'$ *be an ideal, and denote by* $J^e$ *its extension to* $K(Y)[[X]]'$. *Then a Gröbner basis of* $J \subset K[Y][[X]]'$ *w.r.t the term order* $>_x$ *on* $[X]$ *is a Gröbner basis of* $J^e \subset K(Y)[[X]]'$ *with respect to the term order* $>_x$ *on* $[X]$, *hence* $\mathrm{in}_{>_x}(J)^e = \mathrm{gr}_{>_x}(J^e)$.

*Proof.* Similar to [7, Corollary 3.7] and [34, Prop. 3.4]. □

**Lemma 3.6.10.** *For any ideal* $J \subset K[Y][[X]]'$, *we have that*

$$\mathrm{gr}(J) = \mathrm{gr}(\mathop{\mathrm{in}}_{>_x}(J)).$$

*Proof.* The filtration on $K[Y][[X]]'$ induced by $>$ is a refinement of the filtration induced by $>_x$. □

**Lemma 3.6.11.** *If* $J \subset K[Y][[X]]'$ *is a bi-homogeneous,* $\mathcal{M}$*-locally finitely generated ideal, then a* $\mathcal{M}$*-homogeneous Gröbner basis* $G$ *of* $J$ *with respect to* $>$ *is also a Gröbner basis of* $J$ *with respect to* $>_x$.

*Proof.* It follows from the discussion in the appendix that a $>$-Gröbner basis of a bi-homogeneous, $\mathcal{M}$-locally finitely generated ideal in $K[Y][[X]]'$ is a generating set of the ideal; hence it follows that if two bi-homogeneous, $\mathcal{M}$-locally finitely generated ideals $A \subset B$ have the same initial ideal with respect to $>$, then they are equal. Now apply this to the ideal $A$ generated by $\mathrm{in}_{>_x}(G)$ and to the ideal $B = \mathrm{in}_{>_x}(J)$. Then $A \subset B$. By Lemma 3.6.10 , we have that $\mathrm{gr}_{>}(B) = \mathrm{gr}_{>}(\mathrm{in}_{>_x}(J)) = \mathrm{gr}_{>}(J)$, and since $G$ is a $>$-Gröbner basis for $J$, we get that $\mathrm{gr}_{>}(A) = \mathrm{gr}_{>}(J)$. Therefore, $A = B$, that is, $G$ is a $>_x$-Gröbner basis of $J$. □

The above result resembles the well-known result that a Gröbner basis of an ideal in $K[x_1, \ldots, x_n]$, w.r.t a term order that is *degree-compatible*, that is, refines the total-degree partial order, is also a *Macaulay basis* of the ideal, that is, the set of homogeneous components of maximal total degree of the basis generates the ideal of all homogeneous components of maximal degree of elements in the ideal.

**Theorem 3.6.12.** *For the pure generic ideal* $I \subset K[Y][[X]]'$*, the following assertions hold:*

*(i)* $\mathrm{gr}(I^e)$ *is generated by a finite number of monomials in* $\mathcal{M}^r$*,*

*(ii)* $\mathrm{gr}(I^e)^c$ *is generated by a finite number of monomials in* $\mathcal{M}^r$*,*

*(iii)* $\mathrm{in}_{>_x}(I)$ *is generated by a finite number of elements of the form* $pm$*,* $p \in K[y_1, \ldots, y_{t_r}]$*,* $m \in \mathrm{gr}(I)^e \cap \mathcal{M}^r$*,*

*(iv) The coefficient ideal* $(\mathrm{in}_{>_x}(I) : m) \cap K[Y]$*,* $m \in \mathcal{M}$*, is zero if* $m \notin \mathrm{gr}(I^e)$*, and generated by finitely many* $p_i \subset K[y_1, \ldots, y_{t_r}]$ *otherwise.*

*Proof.* The first assertion follows from Theorem 3.4.2, since $I^e$ is a generic ideal in $K(Y)[[X]]'$.

The second assertion follows trivially from the first.

To prove the third assertion, we note that I has a $>$-Gröbner basis, and hence a $>_x$-Gröbner basis consisting of elements $f$ which have $\mathrm{Lpp}_{>_x}(f) \in \mathcal{M}^r$ and $\mathrm{Lpp}_{>_y}(\mathrm{lc}_{>_x}(f)) \in K[y_1, \ldots, y_{t_r}]$. By the construction of I, we must in fact have that $\mathrm{lc}_{>_x}(f) \in K[y_1, \ldots, y_{t_r}]$.

The fourth assertion follows from the preceding ones. We know that

$$\mathop{\mathrm{in}}_{>_x}(I) = (p_1 m_1, \ldots, p_\nu m_\nu),$$

that $p_i \in K[y_1, \ldots, y_{t_r}]$, and that $m_\nu \in \mathrm{gr}(I^e) \cap \mathcal{M}^r$. Therefore, for any $m \in \mathcal{M}$, $(\mathrm{in}_{>_x}(I) : m) \cap K[Y]$ is generated as a $K[Y]$-ideal by all $p_i$ for those $i$ such that $m_i | m$. □

### *3.6.2 Examples*

Assume temporarily that $\mathrm{char}(K) \neq 2, 3$. For the calculation of $\mathrm{in}_{>_x}(J)$ we have used Lemma 3.6.3 and Lemma 3.6.11: we calculate a Gröbner basis of J with respect to the degrevlex term order (this gives that same result as $>$, by Lemma 3.6.3) and then extract the $>_x$-leading terms.

**Example 3.6.13.** Let J be the pure generic ideal of Example 3.5.14. Then

$$\mathrm{gr}(J^e) = (x_1^2, x_1 x_2, x_2^3)$$

and

$$\begin{aligned}\operatorname*{in}_{>_x}(J) = \Big(&x_1^2\alpha_{11},\ x_1^2\beta_{11},\ x_1x_2(\alpha_{11}\beta_{12}-\alpha_{12}\beta_{11}),\\ &x_1x_2^2(\beta_{1,1}\alpha_{2,2}-\alpha_{1,1}\beta_{2,2}),\ x_1x_2^2(\alpha_{1,1}\alpha_{2,2}\beta_{1,2}-\alpha_{1,1}\alpha_{1,2}\beta_{2,2}),\\ &x_2^3(\beta_{1,1}^2\alpha_{2,2}^2-\beta_{1,1}\alpha_{1,2}\alpha_{2,2}\beta_{1,2}+\alpha_{1,1}\alpha_{2,2}\beta_{1,2}^2+\beta_{1,1}\alpha_{1,2}^2\beta_{2,2}\\ &\qquad -2\alpha_{1,1}\beta_{1,1}\alpha_{2,2}\beta_{2,2}-\alpha_{1,1}\alpha_{1,2}\beta_{1,2}\beta_{2,2}+\alpha_{1,1}^2\beta_{2,2}^2)\Big)\end{aligned}$$

## 3.7 Initial ideals of finitely generated, homogeneous ideals in $K[Y][[X]]'$

Having examined the initial ideals of generic homogeneous, finitely generated ideals in $R'$, we are ready to turn to the study of the initial ideals of arbitrary homogeneous, finitely generated ideals in $R'$. In particular, the following question is of great interest:

**Question 3.7.1.** *Let $J$ be a homogeneous, finitely generated ideal in $R'$, and let* $\mathrm{gr}(J)$ *denote its initial ideal, with respect to the graded reverse lexicographic order. Is* $\mathrm{gr}(J)$ *finitely generated?*

We will not be able to answer Question 3.7.1 in this paper, but we will nonetheless endeavour to shed some light upon it, using our knowledge of initial ideals of generic ideals, and the fact that finitely generated, homogeneous ideals in $R'$ may in some sense be regarded as the "specialization" of the corresponding pure generic ideal.

**Definition 3.7.2.** We say that a $\mathcal{M}$-homogeneous, finitely generated ideal, in $R'$ or in $K[Y][[X]]'$, is of type $d_1, \ldots, d_r$ if it can be generated by forms of these degrees. It is of minimal type $d_1, \ldots, d_r$ if it can be generated minimally by forms of these degrees.

### 3.7.1 The concept of specialization

**Definition 3.7.3.** A $K$-algebra homomorphism $\phi : K[Y][[X]]' \to R'$ is called a *specialization* if $\phi(K[Y]) \subset K$, and if $\phi(x_i) = x_i$ for all $i$. The specialization is said to be *good* if $\phi(T)$ is algebraically independent over the prime field $Q$ for any $T \subset Y$.

**Example 3.7.4.** A good specialization may map a set $V \subset K[Y]$ that is algebraically independent over $Q$ to a subset of $K$ that is *not* algebraically independent over $Q$. For instance, suppose that $A = \{ a_i | i \in \mathbb{N}^+ \}$ is algebraically dependent, whereas

$$V = \{ a_i + \xi_i - \theta(\xi_i) | i \in \mathbb{N}^+ \} \subset K[Y]$$

is algebraically independent. Then if $\theta$ is any (good or otherwise) specialization then $\phi(V) = A$. As an example, if $Q$ is the field of rational numbers, then we may take $a_i = i$, $\xi_i = y_i$.

Clearly, if $\theta$ is a good specialization of $I$, then $\theta(I)$ is a generic ideal in $R'$. In this case, we also have that

$$\mathrm{gr}(\theta(I)) = \theta(\underset{>_x}{\mathrm{in}}(I)) = \mathrm{gr}(I^e) \cap R',$$

where the extension is to $K(Y)[[X]]'$.

**Lemma 3.7.5.** *If* $I \subset K[Y][[X]]'$ *is a pure generic ideal of minimal type* $d_1, \dots, d_r$, *and if* $J \subset R'$ *is a finitely generated, homogeneous ideal of type* $d_1, \dots, d_r$, *then there exists some specialization* $\phi$ *such that* $\phi(I) = J$.

We can now reformulate Question 3.7.1:

**Question 3.7.6.** *Let* $I$ *be a pure generic ideal in* $K[Y][[X]]'$, *and let* $\phi$ *be a specialization. Is the monomial ideal* $\mathrm{gr}(\phi(I)) \subset R'$ *finitely generated?*

The following result is a straightforward generalization of the corresponding result in [7, Proposition 3.4] (it also bears some resemblance to [76, Lemma 10.3]). The short proof of that proposition works *mutatis mutandis*.

**Lemma 3.7.7.** *Let* $J$ *be an ideal in* $K[Y][[X]]'$. *For any admissible order* $>_x$ *on* $[X]$, $>_y$ *on* $[Y]$ *and any specialization* $\phi$, *we have that*

$$\phi(\underset{>_x}{\mathrm{in}}(J)) \subset \underset{>_x}{\mathrm{gr}}(\phi(J)).$$

*Similarly,*

$$\phi(\underset{>}{\mathrm{gr}}(J)) \subset \underset{>_x}{\mathrm{gr}}(\phi(J)).$$

*where* $>$ *is the "twisted" product of* $>_x$ *and* $>_y$ *on* $[Y] \oplus [X]$.

*Proof.* We prove only the first assertion, the second is similar.

It is enought to show that each generator of $\phi(\mathrm{in}_{>_x}(J))$ also belongs to $\mathrm{gr}_{>_x}(\phi(J))$. The ideal $\phi(\mathrm{in}_{>_x}(J))$ is generated by all $\phi(\mathrm{in}_{>_x}(f))$ with $f \in J$. For each $f \in J$, either $\phi(\mathrm{in}_{>_x}(f)) = 0$, or else we have that

$$\phi(\underset{>_x}{\mathrm{in}}(f)) = \phi(\underset{>_x}{\mathrm{lc}}(f)\underset{>_x}{\mathrm{Lpp}}(f)) = \phi(\underset{>_x}{\mathrm{lc}}(f))\underset{>_x}{\mathrm{Lpp}}(f) \in \underset{>_x}{\mathrm{gr}}(\phi(J)).$$

□

From now on, we once again assume that $>_x$ and $>_y$ are degrevlex, and that $>$ is their twisted product.

**Theorem 3.7.8.** *Let $\phi : K[Y][[X]]' \to R'$ be a specialization, and let $I$ be a pure generic ideal in $K[Y][[X]]'$. Then $\phi(\mathrm{in}_{>_x}(I))$ is a finitely generated monomial ideal, as is $\phi(\mathrm{gr}_>(I))$.*

*Proof.* The ideals $\mathrm{in}_{>_x}(I)$ and $\mathrm{gr}_>(I)$ are finitely generated ideals in $K[Y][[X]]'$, hence any specialization of them is a finitely generated monomial ideal in $K[[X]]'$. □

By Theorem 3.7.8 and Lemma 3.7.7, we know that $\mathrm{gr}_{>_x}(\phi(I))$ contains the finitely generated monomial ideals $\phi(\mathrm{in}_{>_x}(I))$ and $\phi(\mathrm{gr}_>(I))$. $R'$-ideals may be regarded as $R'$-modules; therefore, we can form the quotient modules $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{in}_{>_x}(I))}$ and $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{gr}_>(I))}$. Now, an ideal is a finitely generated module iff it is a finitely generated ideal, and the quotient of a module $A$ with a finitely generated module $B$ is finitely generated iff $A$ is finitely generated. Therefore, we conclude:

**Proposition 3.7.9.** *Let $I$ be a pure generic ideal of $K[Y][[X]]'$, and let*

$$\phi : K[Y][[X]]' \to R'$$

*be a specialization. Then the following assertions are equivalent:*

*(i) The $R'$-ideal $\mathrm{gr}_{>_x}(\phi(I))$ is a finitely generated ideal,*

*(ii) The $R'$-module $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{in}_{>_x}(I))}$ is finitely generated,*

*(iii) The $R'$-module $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{gr}_>(I))}$ is finitely generated.*

**Question 3.7.10.** *If $I \subset K[Y][[X]]'$ is a pure generic ideal, and*

$$\phi : K[Y][[X]]' \to R'$$

*is a specialization, when are the quotient modules $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{in}_{>_x}(I))}$ and $\frac{\mathrm{gr}_{>_x}(\phi(I))}{\phi(\mathrm{gr}_>(I))}$ non-zero modules? Clearly, if $\phi$ is good, then the quotient modules are zero.*

**Example 3.7.11.** This example was found using the computer algebra program Bergman [4]. Let

$$J = (f, g) \subset K[Y][[X]]'$$

be as in Example 3.5.14, and let $\phi$ be the specialization determined by

$$\phi(\alpha_{ij}) = \begin{cases} 1 & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(\beta_{ij}) = \begin{cases} 1 & \text{if } (i,j) = (1,2), \\ -1 & \text{if } (i,j) = (3,3), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi(J) = (x_1^2, x_1x_2 - x_3^2)$. The initial ideal is

$$\mathrm{gr}(\phi(J)) = (x_1^2, x_1x_2, x_1x_3^2, x_3^4).$$

So, in degree 4 the initial ideal has more minimal generators than does the initial ideal of the corresponding generic ideal $\theta(J)$ ($\theta$ is any good specialization), which is $(x_1^2, x_1x_2, x_2^3)$. From our previous calculations of $\mathrm{gr}_>(J)$ and $\mathrm{in}_{>_x}(J)$ (Example 3.5.14 and Example 3.6.13) we get that the monomial ideal

$$\phi(\underset{>_x}{\mathrm{in}}(J)) \subset \underset{>_x}{\mathrm{gr}}(\phi(J))$$

is equal to $(x_1^2, x_1x_2)$, as is the monomial ideal $\phi(\mathrm{gr}_>(J))$. The quotient module

$$\frac{\mathrm{gr}_{>_x}(\phi(J))}{\phi(\mathrm{in}_{>_x}(J))}$$

is generated by the images of $x_1x_3^2$ and $x_3^4$.

An affirmative answer to Question 3.7.1 and Question 3.7.6 would have far-reaching consequences. On the other hand, if the answer is negative, it would be interesting to find a counter-example. Perhaps Example 3.7.11 may be improved to yield an example of a homogeneous, finitely generated ideal in $R'$, of type $2, 2$ which has a non-finitely generated initial ideal, with respect to the graded reverse lexicographic order. To this end, one would need to find, for any positive $n$, quadratic polynomials $f_n, g_n \in K[x_1, \dots, x_n]$ such that

A) $\rho_{n-1}(f_n) = f_{n-1}$,

B) $\rho_{n-1}(g_n) = g_{n-1}$,

C) For any positive integer $\nu$, there exists a $\nu' \geq \nu$, a positive integer $N$, and a monomial $m$, such that $x_{\nu'} | m$, and for $n \geq N$ we have that $m$ is a minimal monomial generator of $\mathrm{gr}((f_n, g_n)) \subset K[x_1, \dots, x_n]$.

We conclude with the conjecture (this would follow from Conjecture 3.5.5) that for "sufficiently generic" ideals (specializations of generic ideals which take all except a finite number of coefficients to generic values), the initial ideal is finitely generated:

**Conjecture 3.7.12.** *Let* $I \subset K[Y][[X]]'$ *be a pure generic ideal, and let* $\phi$ *be a specialization such that there exists a finite subset* $U$ *of* $Y$ *with the property that* $\phi(W)$ *is algebraically independent over* $Q$ *for any* $W \subset Y \setminus U$*. Then the graded reverse lexicographic initial ideal* $\mathrm{gr}(\phi(I))$ *is finitely generated.*

### 3.7.2 Acknowledgements

I thank Jörgen Backelin, Ralf Fröberg and Torsten Ekedahl for useful advice and patient tutoring. Many of the proofs are due to them (that is not to say that anyone but myself is to blame for possible errors!) as is many of the ideas used in this paper.

## 3.8 Appendix: Gröbner bases in $K[Y][[X]]'$

In Lemma 3.6.11, we claimed that if $J \subset K[Y][[X]]'$ is a bi-homogeneous, $\mathcal{M}$-locally finitely generated ideal, and if $>$ is the "twisted product" of $>_x$ on $[X]$ and $>_y$ on $[Y]$, and if $G$ is a $>$-Gröbner basis of $J$, then $G$ generates $J$. This will follow if we show that any element $f \in J$ can be written as a $>$-admissible combination of elements in $G$.

We also claimed earlier that $>$-leading power products of elements in $K[Y][[X]]'$ may be defined. The latter assertion can be strengthened: if we regard $K[Y][[X]]'$ as a subring of $K[[Y \cup X]]'$, then we may in fact define leading power products with respect to $>$ of all elements in the larger ring. This is so, because any subset of $Y \cup X$ has a maximal element, and because of the theorem below. Note that since $X, Y$ are countable, $X \cup Y$ is is bijective correspondance with $X$, and $[X \cup Y]$ is in bijective correspondance with $[X] = \mathcal{M}$.

**Theorem 3.8.1.** *Let* $>$ *be a total order on* $\mathcal{M}$ *which is compatiple with the monoid operation, and is such that any subset* $S \subset \mathcal{M}$ *has a maximal element with respect to* $>$*. Then*

$$f \in R' \setminus \{0\} \implies \mathrm{Mon}(f) \text{ contains a maximal element.}$$

*Proof.* If we denote by $<$ the opposite relation of $>$, we shall prove: $<$ is a well-order restricted to any $\mathcal{M}_d$ iff $<$ restricted to $\mathcal{M}_1$ is. One implication is immediate, so we concentrate on the other: we assume that $<$ is a well-order on $\mathcal{M}_1$, and fix a $d$. We assume inductively that for $k < d$, $<$ is a well-order on $\mathcal{M}_k$, and show that it must be a well-order on $\mathcal{M}_d$.

Let

$$m_1 < m_2 < m_3 < \dots \tag{3.6}$$

be a strictly increasing sequence in $\mathcal{M}_d$, finite or infinite. Write

$$m_j = \prod_{i=1}^{d} x_{\alpha_{j,i}} \tag{3.7}$$

with $x_{\alpha_{j,1}} \geq x_{\alpha_{j,2}} \geq \cdots \geq x_{\alpha_{j,d}}$, and note that it follows from a generalization of [75, Lemma 2.3] that $x_{\alpha_{j+1,1}} > x_{\alpha_{j,d}}$; in particular, $x_{\alpha_{j,1}} > x_{\alpha_{1,d}}$ whenever $j > 1$. Thus, the sequence $\{x_{\alpha_{j,1}}\}_{j=1}^{\infty}$ is bounded from below.

Now, from our hypothesis, the sequence $\{x_{\alpha_{j,1}}\}_{j=1}^{\infty}$ may not contain an infinite, strictly increasing subsequence. Neither can it contain an infinite, decreasing sequence: if it does, then corresponding subsequence of the $\bar{m}_j$'s, where $\bar{m}_j = \prod_{i=2}^{d} x_{\alpha_{j,i}}$, must be strictly increasing; we then get an infinite, strictly increasing sequence in $\mathcal{M}_{d-1}$, contradicting the induction hypothesis.

We want to show that (3.6) is finite: we will do this by showing that the sequence $\{x_{\alpha_{j,1}}\}$ must be finite. A simple, quite general lemma will conclude the proof. □

**Lemma 3.8.2.** *Let $\{a_j\}_{j=1}$ be a (finite or infinite) sequence in a totally ordered set. Suppose that the sequence contains no increasing, infinite subsequence, and no infinite, decreasing subsequence (and consequently no infinite constant subsequence). Then the sequence is finite.*

*Proof.* First, note that it is enough to prove that the sequence contains no countably infinite subsequence, hence we may assume the sequence to be countable. Then, it is clear that we may embed the sequence in the real intervall $[0, 1]$ via an order-preserving mapping $\phi$. So, we may assume that we have a (countable or finite) sequence in $[0, 1]$, containing no infinite increasing or decreasing subsequence. We claim that the sequence can contain no limit point $y$. Suppose that it does. Then, the sequence contains an infinite subsequence converging to $y$. If we can show that this sequence must contain either an infinite increasing or an infinite decreasing subsequence, we have a contradiction. Now, note that the sequence must contain either infinitely many elements $\geq y$, or infinitely many elements $\leq y$. Assume the latter to be the case. Then, since the sequence contains no infinite constant subsequence, and since for any $n$ there exists an element in the sequence that is contained in $[y - 2^{-n}, y]$, we get immediately the existence of an infinite, strictly increasing subsequence. This is a contradiction. Therefore, the sequence contains no limit points.

Now, applying the Bolzano-Weirstrass theorem, we get that the original sequence must be finite. □

The following lemma generalizes [75, Remark 3.1]. For a monomial $m \in [Y] \oplus [X]$, of the form $tt'$, with $t \in [Y]$, $t' \in [X]$, we define $\mathrm{maxsupp}_x(m)$ as the highest index $i$ such that $x_i \,|t'$. We define $\mathrm{maxsupp}_y(m)$ as the highest index $j$ such that $y_j \,|t$.

**Lemma 3.8.3.** *If* $t_x \in [X]$, $t_y \in [Y]$, $f \in K[Y][[X]]'$, *and*

$$N_x = \underset{x}{\mathrm{maxsupp}}(\mathrm{Lpp}(f))$$
$$N_y = \underset{y}{\mathrm{maxsupp}}(\mathrm{Lpp}(f))$$

*then* $\mathrm{Lpp}(f) \,|t_x t_y$ *iff* $\mathrm{Lpp}(f) \left|t_x' t_y'\right.$, *where* $t_x'$ *denotes the sub-word of* $t_x$ *that is obtained by replacing any occurance of* $X$*-variables not in* $\{x_1, \dots, x_{N_x}\}$ *with 1, and similarly,* $t_y'$ *denotes the sub-word of* $t_y$ *that is obtained by replacing any occurance of* $Y$*-variables not in* $\{y_1, \dots, y_{N_y}\}$ *with 1.*

*If* $F \subset K[Y][[X]]'$ *is a set such that both*

$$S_x = \sup\left\{ \underset{x}{\mathrm{maxsupp}}(\mathrm{Lpp}(f)) \middle| f \in F \right\}$$

*and*

$$S_y = \sup\left\{ \underset{y}{\mathrm{maxsupp}}(\mathrm{Lpp}(f)) \middle| f \in F \right\}$$

*are finite (in particular, if* $F$ *is finite), then* $t_x t_y$ *is divisible by some* $\mathrm{Lpp}(f)$ *with* $f \in F$ *iff* $t_x' t_y'$ *is, where* $t_x'$ *denotes the* $\{x_1, \dots, x_{S_x}\}$*-part of* $t_x$*, and* $t_y'$ *denotes the* $\{y_1, \dots, y_{S_y}\}$*-part of* $t_y$.

It is now easy to see that we may modify [75, Proposition 3.2] to show that we may calculate $>$-normal forms of elements in $K[Y][[X]]'$ with respect to a finite set of monic elements: we regard $K[Y][[X]]'$ as a polynomial ring on the variables $y_1, \dots, y_{S_y}$ and $x_1, \dots, x_{S_x}$, with coefficients in the domain $D = K[Y^c][[X^c]]'$, where $X^c$and $Y^c$ are the remaining variables. Supposing now that the elements of $F$ are monic, and $h \in K[Y][[X]]'$, we want to calculate normal forms of $h$ with respect to $F$. We can regard $h$ as an element in the polynomial ring on the variables $y_1, \dots, y_{S_y}$ and $x_1, \dots, x_{S_x}$ with coefficients in $D$, calculate normal forms with the aid of the well-known division algorithm for polynomial rings with coefficients in commutative rings, and then "lift" the normal forms in the polynomial ring to normal forms in $K[Y][[X]]'$, just as in [75].

If we calculate the normal form of a bi-homogeneous element $f \in J$ of $\mathcal{M}$-total degree $d$ with respect to a truncated Gröbner basis $G_{\leq d}$ for $J$, then clearly the normal form is zero, and the expression of $f$ as a finite sum of products of elements in $G_{\leq d}$ and homogeneous elements in $K[Y][[X]]'$ is an admissible combination.

## 3.9 Appendix: A Macaulay 2 session

We demonstrate how to compute the initial ideals of pure generic ideals, as in Section 3.5.4, by means of the computer algebra program Macaulay 2 [38]. Recall that we are dealing with the polynomial ring $K[y_{i,m}; x_1, \ldots, x_n]$, where $1 \leq i \leq r$, where $d_1, \ldots, d_r \in \mathbb{N}^+$, and where $m \in \mathcal{M}^n_{d_i}$. We want to compute $\mathrm{gr}(I)$, where $I = (f_1, \ldots, f_r)$ and $f_i = \sum_{m \in \mathcal{M}^n_{d_i}} y_{i,m} m$ are pure generic forms of bi-degree $(1, d_i)$. The term order used is the degrevlex order (by Lemma 3.6.3 this gives the same initial ideal is the "twisted product" of degrevlex on the two groups of variables). A transcript for the computation of Example 3.5.14 is given below.

```
Macaulay 2 - copyright 1996, Daniel R. Grayson and Michael E. Stillman
Factory library from Singular, copyright 1996, G.-M. Greuel and R. Stobbe
Factorization and characteristic sets library, copyright 1996, M. Messollen


i1 = KK = ZZ/31991
o1 = KK
o1 : QuotientRing
i27 = R = KK[a_{1,1}, b_{1,1}, a_{1,2}, a_{2,2},
      b_{1,2}, b_{2,2}, a_{1,3}, a_{2,3}, a_{3,3},
      b_{1,3}, b_{2,3}, b_{3,3},
      x_1..x_3]
o27 = R
o27 : PolynomialRing
i28 = f1 = sum(1..3, i->sum(i..3, j-> a_{i,j}*x_i*x_j))
               2                            2
o28 = a      x   + a      x  x  + a      x   + a      x  x  +
       {1,1}  1     {1,2}  1  2    {2,2}  2     {1,3}  1  3
                                2
    + a      x  x  + a      x
       {2,3}  2  3    {3,3}  3
o28 : R
i29 = f2 = sum(1..3, i->sum(i..3, j-> b_{i,j}*x_i*x_j))
               2                            2
o29 = b      x   + b      x  x  + b      x   + b      x  x  +
       {1,1}  1     {1,2}  1  2    {2,2}  2     {1,3}  1  3
                                2
    + b      x  x  + b      x
       {2,3}  2  3    {3,3}  3
o29 : R
```

```
i30 = I=ideal(f1,f2);
i31 = GB = gb I;
i32 = J = gens GB;
              1       6
o32 : Matrix R  <--- R
i33 = leadTermMatrix J
o33 = | b_{1,1}x_1^2
        a_{1,1}x_1^2
        b_{1,1}a_{1,2}x_1x_2
        b_{1,1}a_{2,2}x_1x_2^2
        a_{1,1}a_{2,2}b_{1,2}x_1x_2^2
        b_{1,1}^2a_{2,2}^2x_2^3 |
              1       6
o33 : Matrix R  <--- R
```

# 4. GENERALIZED HILBERT NUMERATORS


*Abstract*

It is a well-known fact that if $K$ is a field, then the Hilbert series of a quotient of the polynomial ring $K[x_1, \dots, x_n]$ by a homogeneous ideal is of the form $\frac{q(t)}{(1-t)^n}$; we call the polynomial $q(t)$ the *Hilbert numerator* of the quotient algebra.

We will generalize this concept to a class of non-finitely generated, graded, commutative algebras, which are endowed with a surjective "co-filtration" of finitely generated algebras. Then, although the Hilbert series themselves can not be defined (since the sub-vector-spaces involved have infinite dimension), we get a sequence of Hilbert numerators $q_n(t)$, which we show converge to a power series in $\mathbb{Z}[[t]]$. This power series we call the *(generalized) Hilbert numerator* of the non-finitely generated algebra.

The question of determining when this power series is in fact a polynomial is the topic of the last part of this article. We show that quotients of the ring $R'$ by homogeneous ideals that are generated by finitely many monomials have polynomial Hilbert numerators, as have quotients of $R'$ by ideals that are generated by two homogeneous elements. More generally, the Hilbert numerator is a polynomial whenever the ideal is generated by finitely many homogeneous elements, the images of which form a regular sequence under all but finitely many of the *truncation homomorphisms* $\rho_n$.


## *4.1 Introduction*

The ring $R'$, the "largest graded subring" of the power series ring $R = K[[x_1, x_2, x_3, \dots]]$ on countably many indeterminates, with coefficients in a field $K$, was introduced in [75, 76] as a tool for the study of the "stable parts" of the initial ideals of generic ideals. It was demonstrated, that if $r, d_1, \dots, d_r$ are positive integers, if $f_{i,n} \subset K[x_1, \dots, x_n]$, $1 \leq i \leq r$, are generic forms of total degree $d_i$, if

$$I_n = (f_{1,n}, \dots, f_{r,n}) \subset K[x_1, \dots, x_n]$$

is a generic ideal, and if $>$ is i.e the lexicographic term order, then the initial ideals $\mathrm{gr}(I_n)$ "converges" to the initial ideal of the ideal $I \subset R'$, where $I = (f_1, \dots, f_r)$, and each $f_i$ is the "limit" of the $f_{i,n}$'s, namely a "generic form in infinitely many variables".

More generally, we denote by $\rho_n$ the canonical quotient epimorphism

$$R' \twoheadrightarrow R'/A_n \simeq K[x_1, \dots, x_n],$$

where $A_n$ is the ideal generated by all elements in $R' \cap K[[x_{n+1}, x_{n+2}, \dots]]$ with zero constant term. Let $J \subset R'$ be a so-called locally finitely generated ideal, that is, a homogeneous ideal that can be generated by a set of homogeneous elements, containing only finitely many elements of each given total degree. Then the initial ideals $\mathrm{gr}(\rho_n(J))$ converge to $\mathrm{gr}(J)$.

Returning to the generic ideals $I_n$, we see that the Hilbert series of $K[x_1, \dots, x_n]/I_n$ is equal to

$$(1-t)^{-n} \prod_{i=1}^{r} (1-t)^{d_i}.$$

If we call this Hilbert series $H_n(t)$, then

$$\lim_{n \to \infty} (1-t)^n H_n(t) = \prod_{i=1}^{r} (1-t)^{d_i}.$$

It seems natural to associate to the graded algebra $R'/I$, for which no Hilbert series can be calculated (the graded parts have infinite dimension as K-vector spaces), the "Hilbert numerator" $\prod_{i=1}^{r}(1-t)^{d_i}$.

For the locally finitely generated ideal $J$ , we can, as it happens, make a similar construction: if we write the Hilbert series of $K[x_1, \dots, x_n]/\rho_n(J)$ as $(1-t)^{-n} q_n(t)$, where $q_n(t)$ is a polynomial, then the $q_n$'s converge to a power series $q(t) \in \mathbb{Z}[[t]]$ which we define to be the Hilbert numerator of $R'/J$.

We give an example of an ideal where the Hilbert numerator is not a polynomial. The question of this can happen for finitely generated homogeneous ideals is left unanswered, but we do prove that ideals generated by two elements or less have a polynomial Hilbert numerator, as have ideals generated by finitely many monomials. We give a sufficient condition, based on the distributivity of the lattice generated by the principal ideals on the generators, for a finitely generated ideal to have a polynomial Hilbert numerator.

## 4.2 Preliminaries

We summarize briefly some of the more important notations used in this article; they are described in full detail in [75, 76]. Rings, similar to the rings $R$ and $R'$ introduced below, have been studied in [50, 64, 69].

Let $K$ be any field, and let $R'$ be the smallest $K$-subalgebra of $R = K[[x_1, x_2, x_3, \dots]]$ that contains all homogeneous elements. For $f \in R$, we denote by $|f|$ the total degree of $f$ (we have that $|f| < \infty$ iff $f \in R'$). There is a natural filtration on $R$ given by

$$\mathcal{T}^k R = \{ f \in R \mid |f| \leq k \}.$$

This restricts to a filtration on $R'$, as well as on any ideal $I$ in $R'$. Furthermore, $R'$ may be viewed as the graded associated ring to $R$ with respect to this filtration, and is therefore an $\mathbb{N}$-graded ring.

If $I$ is homogeneous, we henceforth denote by $I_d$ the set of homogeneous elements in $I$ of total degree $d$. This contrast with our notations in the introduction, where $\{I_n\}_{n=1}^{\infty}$ was an indexed set of ideals in different polynomial rings.

Let $I$ be a homogeneous, locally finitely generated ideal of $R'$; that is, $I$ has a homogeneous generating set that contains only finitely many element of a given total degree. Expressed slightly differently: $I$ is homogeneous and

$$\forall d \in \mathbb{N}^+ : \quad \dim_K \left( \frac{I_d}{\sum_{j=1}^{d} R'_j I_{d-j}} \right) < \infty.$$

In particular, homogeneous and finitely generated ideals are locally finitely generated.

Denote by $\mathcal{M}$ the free commutative monoid on $\{x_1, x_2, x_3, \dots\}$. Let $>$ be an admissible order on $\mathcal{M}$, that is, $>$ is a total order that makes $(\mathcal{M}, >)$ into an ordered monoid; furthermore we demand that 1 is the smallest element, and that $x_1 > x_2 > x_3 > \cdots$. By [75, Theorem 5.12], every non-empty subset of $\mathcal{M}$ such that the sum of its elements is an element in $R'$ has a maximal element with respect to $>$. We can thus define the *leading power product* $\mathrm{Lpp}(f) \in \mathcal{M}$ for any $f \in R'$, and also associate to $I$ the monomial ideal $\mathrm{gr}(I)$ that is generated by all leading power products of elements in $I$. It is proved in [75] that this ideal is also locally finitely generated. This is done by a Gröbner Bases theory for $R'$ which extends the Gröbner Bases theory for polynomial rings over fields [18, 21, 22, 11, 59, 72].

For any positive integer $n$, the *truncation* $\rho_n(I)$ is an ideal of the polynomial ring $K[x_1, \dots, x_n]$. It is defined as the image of $I$ under the truncation homomorphism

$$\begin{aligned} \rho_n : R' &\to K[x_1, \dots, x_n] \\ \sum_{m \in \mathcal{M}} c_m m &\mapsto \sum_{m \in \mathcal{M}^n} c_m m \end{aligned}$$

where $\mathcal{M}^n$ is the subsemigroup of $\mathcal{M}$ that is generated by $\{x_1, \dots, x_n\}$. Clearly $\varinjlim \mathcal{M}^n = \mathcal{M}$, and we can define a function $\mathrm{maxsupp} : \mathcal{M} \to \mathbb{N}$ by associating to each monomial $m$ the minimal $n$ such that $m \in \mathcal{M}^n$. For a non-negative integer $d$, we define $\mathcal{M}_d$ to be the subset of $\mathcal{M}$ consisting of all monomials of total degree (word length) $d$. If

$$R' \ni f = \sum_{m \in \mathcal{M}_d} c_m m,$$

we say that $f$ is a *form* of degree $d$; if in addition no $c_m$ belong to the prime field $Q$ of $K$, and if the $c_m$'s are algebraically independent over $Q$ in the sense of [84], we say that $f$ is a *generic form*. Note that the truncation $\rho_n(f)$ of a generic form $f$ is a generic form in $K[x_1, \ldots, x_n]$, so this generalizes the ordinary definition. We say that an ideal I is a *generic* ideal if it is generated by generic forms, such that no coefficient occurs in two different forms, and such that the union of the coefficients of the forms is algebraically independent over $Q$. Usually, it is understood that the ideal should be generated by finitely many generic forms. A generic ideal in $R'$ truncates to a generic ideal (such as the ones studied in [27, 30] ) in the polynomial ring $K[x_1, \ldots, x_n]$.

The kernel $A_n$ of $\rho_n$ is generated by the elements of

$$K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]] \cap R'$$

with zero constant term, so that we may view the truncation homomorphism as a (split) quotient epimorphism; the splitting is given by the fact that the following composition is the identity on $K[x_1, \ldots, x_n]$:

$$K[x_1, \ldots, x_n] \to R' \to \frac{R'}{A_n} \simeq K[x_1, \ldots, x_n]. \tag{4.1}$$

The above formula shows that the completion of $R'$ with respect to the $A_n$ filtration is isomorphic to the inverse (projective) limit of the inverse system of the polynomial rings $K[x_1, \ldots, x_n]$, with index set the positive integers, and with connecting homomorphisms $K[x_1, \ldots, x_n] \twoheadrightarrow K[x_1, \ldots, x_{n-1}]$ given by the (restricted) truncation homomorphisms. One can show that $R'$ is the subring of this inverse limit which consists of all coherent sequences *with bounded total degree*. We will use this fact in the proof of Proposition 4.6.3 and Lemma 4.6.4.

We will abuse our notations slightly, and use $\rho_n$ to denote all restrictions of the truncation homomorphism $\rho_n : R' \to K[x_1, \ldots, x_n]$. This means that the diagram below commutes:

$$\begin{tikzcd}
R' \arrow[rr, hook] \arrow[dr, "\rho_{n+1}"] \arrow[ddr, "\rho_n"'] & & \varprojlim K[x_1, \ldots, x_n] \arrow[dl, "\rho_{n+1}"'] \arrow[ddl, "\rho_n"] \\
& K[x_1, \ldots, x_{n+1}] \arrow[d, "\rho_n"] & \\
& K[x_1, \ldots, x_n] &
\end{tikzcd}$$

The admissible order $>$ restricts to $\mathcal{M}^n$, hence we may form the initial ideal

$$\mathrm{gr}(\rho_n(I)) \subset K[x_1, \ldots, x_n].$$

A large part of [76] is devoted to the relation between $\mathrm{gr}(\rho_n(I))$ and $\rho_n(\mathrm{gr}(I))$ and the initial ideal $\mathrm{gr}(I)$ itself. We recall the main results:

**Theorem 4.2.1 (Degree-wise approximation of initial ideals).** *If $J$ is a locally finitely generated ideal in $R'$, then for all total degrees $d$ we have that ($\cdot^e$ denoting the extension of ideals to $R'$)*

$$L(d,n) := \mathcal{T}^d \rho_n(\mathrm{gr}(J))^e \subset \mathcal{T}^d \,\mathrm{gr}(\rho_n(J))^e =: R(d,n).$$

*Furthermore, there exists integers $N(d)$, which we call "the necessary number of active variables up to degree $d$", and integers $\hat{N}(d)$, which we call "the sufficient number of active variables up to degree $d$", such that:*

*(i) If $n < N(d)$ then*

$$L(d,n) \subsetneq \mathcal{T}^d \,\mathrm{gr}(J)$$
$$R(d,n) \not\supset \mathcal{T}^d \,\mathrm{gr}(J)$$

*(ii) If $N(d) \leq n < \hat{N}(d)$ then*

$$L(d,n) = \mathcal{T}^d \,\mathrm{gr}(J)$$
$$R(d,n) \supset \mathcal{T}^d \,\mathrm{gr}(J)$$

*(iii) If $\hat{N}(d) \leq n$ then*

$$L(d,n) = \mathcal{T}^d \,\mathrm{gr}(J)$$
$$R(d,n) = \mathcal{T}^d \,\mathrm{gr}(J)$$

**Corollary 4.2.2.** *If $J$ is a locally finitely generated ideal of $R'$, then the following are equivalent:*

*(i) $\mathrm{gr}(J)$ is finitely generated,*

*(ii) $\mathrm{gr}(\rho_n(J))^e$ stabilize when $n$ tends to infinity.*

*Furtermore, if the equivalent conditions hold, then $J$ is finitely generated.*

Note that the stable value of $\mathrm{gr}(\rho_n(J))^e$ must be equal to the stable value of $\rho_n(\mathrm{gr}(J))^e$ which must be equal to $\mathrm{gr}(J)$.

## 4.3 Definition of Hilbert numerators for locally finitely generated ideals

Since the restricted ideals $\rho_n(I)$, with $n$ a positive integer, are homogeneous ideals of the corresponding polynomial rings $K[x_1, \ldots, x_n]$, we may define the quotient algebras

$$U_n = \frac{K[x_1, \ldots, x_n]}{\rho_n(I)}$$

and the Hilbert series $\mathrm{Hilb}_{U_n}(t)$. We define the polynomial $q_n(t) \in \mathbb{Z}[t]$ by $\mathrm{Hilb}_{U_n}(t) = \frac{q_n(t)}{(1-t)^n}$. The idea is then to study $U = R'/I$ by means of the approximations provided by the "co-filtrations" $U \twoheadrightarrow U_n$ given by the following diagram:

$$\begin{tikzcd}
R' \arrow[r, two heads] \arrow[dr, two heads] & K[x_1, \ldots, x_n] \arrow[r, two heads] & U_n = \frac{K[x_1,\ldots,x_n]}{\rho_n(I)} \\
& U = \frac{R'}{I} \arrow[ur, dotted, two heads] &
\end{tikzcd}$$

Now put

$$V_n = \frac{K[x_1, \ldots, x_n]}{\mathrm{gr}(\rho_n(I))}.$$

It is a well know fact that $\mathrm{Hilb}_{U_n}(t) = \mathrm{Hilb}_{V_n}(t)$, regardless of the order $>$.

**Lemma 4.3.1.** *If* $\mathrm{gr}(I)$ *is finitely generated, then there exists an* $N$ *such that, for* $n \geq N$*, we have that*

$$V_n = \frac{K[x_1, \ldots, x_n]}{\mathrm{gr}(\rho_n(I))} = \frac{K[x_1, \ldots, x_N, x_{N+1}, \ldots, x_n]}{\rho_N(\mathrm{gr}(I))^e} \simeq$$
$$\simeq \frac{K[x_1, \ldots, x_N]}{\rho_N(\mathrm{gr}(I))}[x_{N+1}, \ldots, x_n], \quad (4.2)$$

*where the extension is with respect to the natural inclusion*

$$K[x_1, \ldots, x_N] \subset K[x_1, \ldots, x_N, x_{N+1}, \ldots, x_n].$$

*Proof.* It follows from Corollary 4.2.2 that there exists an $N$ such that for $n \geq N$,

$$\mathrm{gr}(I) = \mathrm{gr}(\rho_n(I))^e = \rho_n(\mathrm{gr}(I))^e,$$

where the extension is with respect to the inclusion $K[x_1, \ldots, x_n] \hookrightarrow R'$. From this, (4.2) follows. □

In general, $\mathrm{gr}(I)$ is not finitely generated, and

$$\mathrm{gr}(I) \neq \mathrm{gr}(\rho_n(I))^e \neq \rho_n(\mathrm{gr}(I))^e.$$

Still, from Theorem 4.2.1 it follows that for each total degree $d$, there is an integer $N$ (depending on $d$) such that for $n \geq N$ we have that

$$\mathcal{T}^d \mathrm{gr}(I) = \mathcal{T}^d \mathrm{gr}(\rho_n(I))^e = \mathcal{T}^d \rho_n(\mathrm{gr}(I))^e,$$

where the extension is with respect to the inclusion $K[x_1, \ldots, x_n] \hookrightarrow R'$. We get that

$$\mathcal{T}^d V_n \simeq \mathcal{T}^d \left( \frac{K[x_1, \ldots, x_N]}{\rho_N(\mathrm{gr}(I))} [x_{N+1}, \ldots, x_n] \right)$$

as $K$-vector spaces (the quotients inherit the total-degree filtration $\mathcal{T}$ since they are homogeneous). Since

$$\mathrm{Hilb}_{K[x_{N+1}, \ldots, x_n]}(t) = (1-t)^{-(n-N)},$$

it follows that the coefficients of the power series $\mathrm{Hilb}_{V_n}(t)$ and

$$(1-t)^{-(n-N)} \mathrm{Hilb}_{V_N}(t)$$

coincide up to degree $d$. This relates the "Hilbert Numerators" in the following way:

$$\begin{aligned} \frac{q_n(t)}{(1-t)^n} &\equiv \frac{q_N(t)}{(1-t)^N} \cdot \frac{1}{(1-t)^{(n-N)}} \mod \left(t^{d+1}\right) \\ &\equiv \frac{q_N(t)}{(1-t)^n} \mod \left(t^{d+1}\right) \end{aligned}$$

Thus, if $n' \geq n \geq N$, the power series $\frac{q_n(t)}{(1-t)^n}$ and $\frac{q_{n'}(t)}{(1-t)^n}$ coincide (coefficient-wise) up to degree $d$. We can regard the polynomials $q_n(t)$ and $q_{n'}(t)$, formally elements in $\mathbb{Z}[t]$, as power series in $\mathbb{Z}[[t]]$. The next lemma shows, that they must coincide up to degree $d$.

**Lemma 4.3.2.** *If* $P(t), Q(t) \in \mathbb{Z}[[t]]$ *and if* $\frac{P(t)}{(1-t)^n}$ *and* $\frac{Q(t)}{(1-t)^n}$ *coincide coefficient-wise up to degree* $d$, *then* $P(t)$ *and* $Q(t)$ *coincide coefficient-wise up to degree* $d$.

*Proof.* Since $(1-t)^n \in \mathbb{Z}[[t]]$ is a unit for all $n$, we have that

$$t^d \,\middle|\, \big(P(t) - Q(t)\big) \iff t^d \,\middle|\, \frac{P(t) - Q(t)}{(1-t)^n}.$$

□

**Theorem 4.3.3.** *As $n$ tends to infinity, the polynomials $q_n$ converges in $\mathbb{Z}[[t]]$ (with the $(t)$-adic topology) to a power series $q \in \mathbb{Z}[[t]]$.*

*Proof.* The ring $\mathbb{Z}[[t]]$ is a complete Hausdorff space with respect to the $(t)$-adic topology (see [3, Chapter 10]). By Lemma 4.3.2, $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Z}[[t]]$. □

**Definition 4.3.4.** We call the power series $q$ of Theorem 4.3.3 the *generalized Hilbert numerator* for the graded algebra $U = R'/I$ . We write

$$\mathrm{HilbNum}_U(t) = q(t).$$

**Lemma 4.3.5.** *The Hilbert numerators of $U = R'/I$ and of $V = R'/\operatorname{gr}(I)$ coincide.*

*Proof.* This follows from Theorem 4.2.1. □

Note that we can not hope to generalize the Hilbert series of the restricted algebras $U_n$ to a "generalized Hilbert series" for $U$, since in general the degree $d$ part of the graded algebra $U$ is an infinite-dimensional $K$-vector space.

**Example 4.3.6.** Let $J \subset R'$ be a generic ideal, generated by a quadratic and a cubic form. Let $I$ be the initial ideal, with respect to the lexicographic order, of $J$; then by computer calculations one may convince oneself that

$$I = \left(x_1x_2x_3 \ldots x_{a-1}x_a^2,\ x_1x_2x_3 \ldots x_{b-2}x_b^6;\quad a \geq 1,\ b \geq 2\right).$$

By Lemma 4.3.5 we have that $\mathrm{HilbNum}_{\frac{R'}{I}}(t) = \mathrm{HilbNum}_{\frac{R'}{J}}(t)$; as we shall see in Proposition 4.4.4 and Lemma 4.4.5, $\mathrm{HilbNum}_{\frac{R'}{J}}(t) = 1 - t^2 - t^3 + t^5$. We now study the polynomials $q_n(t)$ associated to $\frac{R'}{I}$. We use the notation $S_v := \frac{K[x_1,\ldots,x_v]}{\rho_v(I)}$.

$$\begin{aligned}
\mathrm{HilbNum}_{S_2}(t) &= 1 - t^2 - t^3 + t^4 - t^6 + t^7 \\
\mathrm{HilbNum}_{S_3}(t) &= 1 - t^2 - t^3 + 2t^5 - 2t^6 + 2t^8 - t^9 \\
\mathrm{HilbNum}_{S_4}(t) &= 1 - t^2 - t^3 + t^5 + t^6 - 3t^7 + 2t^8 + 2t^9 - 3t^{10} + t^{11} \\
\mathrm{HilbNum}_{S_5}(t) &= 1 - t^2 - t^3 + t^5 + t^7 - 4t^8 + 5t^9 - 5t^{11} + 4t^{12} - t^{13} \\
\mathrm{HilbNum}_{S_6}(t) &= 1 - t^2 - t^3 + t^5 + t^8 - 5t^9 + 9t^{10} - 5t^{11} - 5t^{12} \\
&\quad + 9t^{13} - 5t^{14} + t^{15} \\
\mathrm{HilbNum}_{S_7}(t) &= 1 - t^2 - t^3 + t^5 + t^9 - 6t^{10} + 14t^{11} - 14t^{12} \\
&\quad + 14t^{14} - 14t^{15} + 6t^{16} - t^{17} \\
&\vdots \\
\mathrm{HilbNum}_{\frac{R'}{I}}(t) &= 1 - t^2 - t^3 + t^5
\end{aligned}$$

Since $\rho_n(I)^e \subset \rho_{n+1}(I)$, where the extension is to $K[x_1, \ldots, x_n, x_{n+1}]$, $S_{n+1}$ may be regarded as a quotient of $S_n[x_{n+1}]$, and hence, the Hilbert numerators of the truncated algebras are decreasing monotonically from $1 - t^2 - t^3 + t^4 - t^6 + t^7$, converging to $1 - t^2 - t^3 + t^5$.

*Remark 4.3.7.* In the above example, the structure of the initial ideal was determined using the computer algebra program Bergman [4]. The Hilbert series calculations were done by means of the computer algebra program Macaulay 2 [38].

## 4.4 Properties of the generalized Hilbert numerator

**Proposition 4.4.1.** *If $I$ is a finitely generated monomial ideal, then*

$$\mathrm{HilbNum}_{R'/I}(t) \in \mathbb{Z}[t].$$

*Proof.* In this case, write $I = (m_1, \ldots, m_r)$ and let

$$N = \max\{\mathrm{maxsupp}(m_1), \ldots, \mathrm{maxsupp}(m_r)\};$$

for $n > N$ we then have that

$$\frac{K[x_1, \ldots, x_n]}{\rho_n(I)} \simeq \frac{K[x_1, \ldots, x_N]}{(m_1, \ldots, m_r)}[x_{N+1}, \ldots, x_n]$$

from which it follows that $q_n(t) = q_N(t)$. Therefore,

$$\mathrm{HilbNum}_{R'/I}(t) = q(t) = q_N(t) \in \mathbb{Z}[t].$$

□

**Corollary 4.4.2.** *If* $\mathrm{gr}(I)$ *is finitely generated, then*

$$\mathrm{HilbNum}_{R'/I}(t) \in \mathbb{Z}[t].$$

**Corollary 4.4.3.** *If* $I$ *is generated by a finite set of polynomials, or equivalently, if* $I$ *is generated in some* $K[x_1, \ldots, x_N]$*, then*

$$\mathrm{HilbNum}_{R'/I}(t) \in \mathbb{Z}[t].$$

*Proof.* In this case, $\mathrm{gr}(I)$ is finitely generated, since $K[x_1, \ldots, x_N]$ is noetherian. □

**Proposition 4.4.4.** *Let* $I$ *be an ideal of* $R'$ *that is generated by homogeneous elements* $f_1, \ldots, f_r$*, which have the property that for all sufficiently large positive integers* $n$*, the sequence* $(\rho_n(f_1), \ldots, \rho_n(f_r))$ *form a regular sequence in* $K[x_1, \ldots, x_n]$*. Then*

$$\mathrm{HilbNum}_{R'/I}(t) = \prod_{i=1}^{r} \left(1 - t^{|f_i|}\right) \in \mathbb{Z}[t].$$

*Proof.* If $n$ is sufficiently large, then (see [53]) we get that

$$\mathrm{Hilb}_{\frac{K[x_1,\ldots,x_n]}{\rho_n(I)}}(t) = \frac{\prod_{i=1}^{r} \left(1 - t^{|f_i|}\right)}{(1-t)^n}.$$

The assertion follows. □

**Lemma 4.4.5.** *Let* $I$ *be a finitely generated generic ideal in* $R'$*. Then*

$$\mathrm{HilbNum}_{R'/I}(t) \in \mathbb{Z}[t].$$

*Proof.* If the generic ideal I is generated by $r$ generic forms $f_1$ to $f_r$, homogeneous with degrees $d_1, \ldots, d_r$ respectively, then $\rho_n(f_1), \ldots, \rho_n(f_r)$ form a regular sequence whenever $n > r$. □

**Example 4.4.6 (Non-polynomial Hilbert Numerators.).** Consider the locally finitely generated ideal $I = (x_1, x_2^2, x_3^3, x_4^4, \ldots)$. For each positive integer $n$, it is clear that $x_1, x_2^2, \ldots, x_n^n$ is a regular sequence in $K[x_1, \ldots, x_n]$, hence each $\frac{K[x_1,\ldots,x_n]}{(x_1,\ldots,x_n^n)}$ is a complete intersection. Thus

$$\mathrm{Hilb}_{\frac{K[x_1,\ldots,x_n]}{\rho_n(I)}}(t) = \frac{\prod_{i=1}^{n}(1-t^i)}{(1-t)^n}$$

and hence $q_n(t) = \prod_{i=1}^{n}(1-t^i)$. It follows that

$$\mathrm{HilbNum}_{R'/I}(t) = \prod_{i=1}^{\infty}(1-t^i).$$

Writing this as a power series $\sum_{n=0}^{\infty} a_n t^n$, we must show that this power series is not a polynomial. That is the topic of the next lemma.

**Lemma 4.4.7.** $Q(t) = \prod_{i=1}^{\infty}(1-t^i) \in \mathbb{Z}[[t]] \setminus \mathbb{Z}[t]$.

*Proof.* Assume, towards a contradiction, that $Q(t)$ is a polynomial. We may then regard it as a polynomial with coefficients in $\mathbb{R}$, and also as a function from $\mathbb{R}$ to $\mathbb{R}$. Then, for $0 \leq x \leq 1$, $N$ a positive integer, we have that

$$0 \leq Q(x) \leq \prod_{i=1}^{N}(1-x^i) = (1-x)^N \prod_{i=1}^{N}\left(1+x+\cdots+x^{i-1}\right) \leq 1.$$

From this, we conclude that $Q(t)$ has a zero of order at least $N$ at $t=1$; this implies in particular, since we know that $Q(t)$ is not identically zero, that the degree of $Q$ is at least $N$. Since $N$ is arbitrary, this is a contradiction. □

## 4.5 When is the generalized Hilbert numerator a polynomial?

In [77] it is proved (by translating some results from [55] to the infinitely-many-variables setting) that the initial ideal, with respect to the degrevlex term order, of a finitely generated generic ideal in $R'$, is finitely generated. One may ask, if any finitely generated, homogeneous ideal in $R'$ has a finitely generated degrevlex initial ideal. If this were to hold true, we could immediately conclude that the Hilbert numerator of any homogeneous, finitely generated ideal in $R'$ must be a polynomial.

However, the reverse implication need not automatically hold: that is, there is no a priori reason to exclude the possibility of homogeneous, finitely generated ideals that have non-finitely generated initial ideals with respect to *any* admissible order, yet have a polynomial Hilbert numerator.

We ask:

**Question 4.5.1.** *Let* $I$ *be a finitely generated (homogeneous) ideal in* $R'$. *Is then* $\mathrm{HilbNum}_{R'/I}(t)$ *a polynomial?*

We will not be able to provide a complete answer to this question, but we shall give affirmative answers for some special cases.

### 4.5.1 Some short exact sequences

We now recall some facts about Hilbert series of homogeneous quotients of polynomial rings.

**Lemma 4.5.2.** *The Hilbert series of* $K[x_1, \dots, x_n]/(f)$*, when* $f \notin K$ *is homogeneous, is*

$$\frac{1 - t^{|f|}}{(1-t)^n}.$$

*Proof.* Use the short exact sequence

$$0 \to K[x_1, \dots, x_n] \xrightarrow{\cdot f} K[x_1, \dots, x_n] \to \frac{K[x_1, \dots, x_n]}{(f)} \to 0$$

and the additivity of $\dim_K$, to conclude that

$$\mathrm{Hilb}_{\frac{K[x_1,\dots,x_n]}{(f)}}(t) = \frac{1}{(1-t)^n} - \frac{t^{|f|}}{(1-t)^n}.$$

□

**Lemma 4.5.3.** *The Hilbert series of* $K[x_1, \dots, x_n]/(f, g)$*, when* $f, g \notin K$ *are homogeneous, is*

$$\frac{1 - t^{|f|} - t^{|g|} + t^{|f|+|g|-|\gcd(f,g)|}}{(1-t)^n}.$$

*Proof.* Use the short exact sequence

$$0 \to \frac{\mathrm{Ann}(\bar{g})}{(f)} \to \frac{K[x_1, \dots, x_n]}{(f)} \xrightarrow{\cdot g} \frac{K[x_1, \dots, x_n]}{(f)} \to \frac{K[x_1, \dots, x_n]}{(f, g)} \to 0 \quad (4.3)$$

□

To apply this this result to the ring $R'$, we need some technical formulas for how gcd's and truncations interact. These formulas are collected in an appendix.

**Proposition 4.5.4.** *If* $f, g \in R' \setminus K$ *are homogeneous, and* $I = (f, g)$*, then*

$$\mathrm{HilbNum}_{R'/I}(t) = 1 - t^{|f|} - t^{|g|} + t^{|f|+|g|-|\gcd(f,g)|}.$$

*Proof.* Combine Lemma 4.5.3 and Corollary 4.6.9. □

### 4.5.2 Lattices of ideals

We would like to extend this result to ideals generated by more than two generators. The trouble is, that Lemma 4.5.3 does not generalize to more than two generators: that is, it is not the case that the Hilbert series of the quotient of $K[x_1, \ldots, x_n]$ by a homogeneous ideal is completely determined by the total degrees of the gcd's of subsets of a generating set.

**Example 4.5.5.** This example is from [6], where it provids an example of an algebra with deviant Poincarè series. We shall concern ourselves instead with its Hilbert series. Let $J$ denote the ideal $(x_1^2, x_2x_3, x_1x_3 + x_2^2)$. Then the generators are pair-wise coprime, so every gcd is 1. Still, $\frac{K[x_1,x_2,x_3]}{J}$ is no complete intersection, and

$$\mathrm{Hilb}_{\frac{K[x_1,x_2,x_3]}{J}}(t) = \frac{1 - 3t^2 + 4t^4 - 2t^5}{(1-t)^3},$$

rather than $\frac{1-3t^2+3t^4-t^6}{(1-t)^3}$, which is the Hilbert series of a quotient of the polynomial ring $K[x_1, x_2, x_3]$ by a an ideal generated by three quadratic forms in a regular sequence.

The exact criteria for when the degrees of the gcd's of the generators determine the Hilbert series is this:

**Theorem 4.5.6.** *Let $I = (f_1, \ldots, f_r)$, $J = (g_1, \ldots, g_r)$ be two homogeneous ideals of $K[x_1, \ldots, x_n]$. Assume that the choosen generators are homogeneous and of total degree $\geq 1$, and that for each subset $S \subset \{1, \ldots, r\}$ we have that $|\gcd(f_{s_1}, \ldots, f_{s_l})| = |\gcd(g_{s_1}, \ldots, g_{s_l})|$.*

*Denote by $L, L'$ the sub-lattices of all ideals of $K[x_1, \ldots, x_n]$ (under the lattice operations $+$ and $\cap$) that is generated by the principal ideals $(f_1), \ldots, (f_r)$, and by $(g_1), \ldots, (g_r)$, respectively. Then the following holds:*

*a) If both $L$ and $L'$ are distributive, then the Hilbert series of $\frac{K[x_1,\ldots,x_n]}{I}$ and $\frac{K[x_1,\ldots,x_n]}{J}$ coincide. Furthermore their Hilbert numerator is a polynom that is given by an explicit formula involving only the gcd-degrees of the various subsets of the $f_i$'s.*

*b) If exactly one of the lattices $L$ and $L'$ distribute, then the Hilbert series of $\frac{K[x_1,\ldots,x_n]}{I}$ and $\frac{K[x_1,\ldots,x_n]}{J}$ differ.*

*Proof.* See [82, 5]. □

Combining this theorem with Corollary 4.6.9 (and Remark 4.6.10), we have the following:

**Proposition 4.5.7.** *Let* $I = (f_1, \dots, f_r)$ *be a homogeneous ideal in* $R'$*, and suppose that the* $f_i$*'s are homogeneous. Suppose furthermore that there exists a positive integer* $N$*, such that, for* $n > N$*, the sub-lattice generated by*

$$\{(\rho_n(f_1)), \dots, (\rho_n(f_1))\}$$

*in the lattice of ideals of* $K[x_1, \dots, x_n]$ *(with lattice operations* $+$ *and* $\cap$*), is distributive. Then,*

$$\mathrm{HilbNum}_{\frac{R'}{I}}(t) \in \mathbb{Z}[t],$$

*and this polynomial is determined by an explicit formula involving only the gcd-degrees of the various subsets of the* $f_i$*'s.*

This proposition provides a new proof of Proposition 4.4.4 and Proposition 4.4.1: the lattice, generated by principal ideals of elements that form a regular sequence, is distributive. So is the the lattice generated by principal ideals generated by monomials. For a proof of this fact, see [82] and [5].

## 4.6 Appendix: The relation between truncation and division

**Lemma 4.6.1.** *If* $f, g \in R'$*,* $f \mid g$ *and* $f, g$ *are not associates, then* $|f| < |g|$*. If* $h \in R' \setminus \{0\}$ *then* $h$ *is a unit iff* $|h| = 0$*, that is, if* $h \in K \setminus \{0\}$*.*

*Proof.* Obvious. □

**Lemma 4.6.2.** *For any* $f \in R'$ *and any positive integer* $n$*, we have that* $|\rho_n(f)| \leq |f|$*. The strict inequality holds iff the homogeneous component of* $f$ *of maximal degree restricts to zero. Thus, equality holds for almost all* $n$*.*

*Proof.* Obvious, since $\rho_n$ is a homogeneous (in fact, multi-homogeneous) $K$-algebra homomorphism. □

**Proposition 4.6.3.** *Let* $f, g \in R'$*. Then* $f \mid g \iff \forall n : \rho_n(f) \mid \rho_n(g)$.

*Proof.* If $g = fh$, then, for all $n$, $\rho_n(g) = \rho_n(f)\rho_n(h)$.

Conversely, assume that for all $n$, there exists $h_n \in K[x_1, \dots, x_n]$ such that $\rho_n(g) = \rho_n(f)h_n$. Then, fixing an $n$, for $\nu > n$ we have that

$$\begin{aligned}
\rho_\nu(g) &= \rho_\nu(f)h_\nu \\
\rho_n(\rho_\nu(g)) &= \rho_n(\rho_\nu(f)\rho_\nu(h_\nu)) \\
\rho_n(g) &= \rho_n(f)\rho_n(h_\nu) \\
\rho_n(g) &= \rho_n(f)h_n
\end{aligned}$$

Since $K[x_1, \ldots, x_n]$ is a domain, these equations imply that $\rho_n(h_\nu) = h_n$. We are using the elementary fact that the multiplicative monoid of a (commutative) domain is cancellative [43, Exercise 7, Section 3.2]. So, the $h_n$'s form a coherent sequence of bounded total degree, yielding an element $h \in R'$ such that $\forall n$ : $\rho_n(h) = h_n$. Then, clearly, $\forall n : \rho_n(h)\rho_n(f) = \rho_n(g)$. It is easy to see that $g = fh$, since the coherent sequence $(h_n)_{n=1}^{\infty}$ corresponds to the quotient of the coherent sequences $(\rho_n(g))_{n=1}^{\infty}$ and $(\rho_n(f))_{n=1}^{\infty}$. □

**Lemma 4.6.4.** *An element* $f \in R'$ *is irreducible iff* $\rho_n(f)$ *is irreducible for almost all* $n$.

*Proof.* By contraposition, we can instead choose to prove the following: $f$ is reducible iff there are infinitely many positive integers $n_i$ such that $\rho_{n_i}(f)$ is reducible.

If $f$ is reducible, that is, if $f = gh$ for some $g, h \in R'$, it follows from Proposition 4.6.3 that $\rho_n(f)$ is reducible for all $n$.

Conversely, suppose that there exists an infinite strictly ascending sequence $n_i \to \infty$ of positive integers, such that $\rho_{n_i}(f)$ is reducible for all $i$. We can, without loss of generality, assume that the sequence consists of all positive integers $\geq N$, where $N = \min\{\, n \,|\, |\rho_n(f)| = |f| \,\}$, since

$$k < n, \quad \rho_n(f) = pq \implies \rho_k(f) = \rho_k(p)\rho_k(q).$$

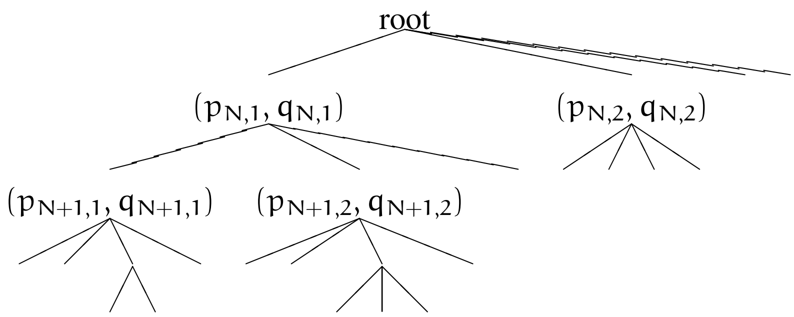
Furthermore, since $K[x_1, \ldots, x_n]$ is an unique factorization domain, where the multiplicative units are contained in $K$, there are (up to multiplication with constants) only finitely many different pairs $(p, q) \in K[x_1, \ldots, x_n]$ such that $\rho_n(f) = pq$. We construct a tree $S$ in the following manner: the vertices "at level $n$" are all pairs $(p, q)$ as above (this subset is denoted by $S_n$). We add one vertex at level $N - 1$, which will be the root of the tree. There is an edge between $(p, q) \in S_n$ and $(p', q') \in S_{n-1}$, that is, $(p, q)$ is a child of $(p', q')$, if $\rho_{n-1}(p) = cp'$ and $\rho_{n-1}(q) = q'/c$ for some $c \in K$. If we construct the tree inductively, we can choose the representatives for $(p, q)$ (under the equivalence $(a, b) \sim (\bar{a}, \bar{b})$ iff $(\bar{a}, \bar{b}) = (\nu a, \nu b)$ for some $\nu \in K$) such that the constant $c$ is always equal to 1. We henceforth assume this.

There are also edges between the root and all vertices at level $N$. As observed above, to every vertex in $S$ there exist a (unique) branch from the root, so $S$ is indeed a tree. It is infinite, but each $S_n$ is finite. The tree $S$ will look something like Fig. 4.1.

We recall that the so-called Königs Lemma [48] states that a countably infinite tree with finite branching contains an infinite branch[1]. Applying Königs lemma to

[1] Equivalent formulations are, that such a tree has a non-empty *body* [56] or that the first infinite ordinal $\omega$ has the *tree property* [49, Chapter IX, Definition 2.13].

***Fig. 4.1:*** *The tree* $S$. *The (ascending) edges are pairs of truncation homomorphisms.*




$S$, we get two infinite sequences $(p_n)_{n=N}^{\infty}$ and $(q_n)_{n=N}^{\infty}$ such that for all $n \geq N$, $\rho_n(f) = p_n q_n$, $\rho_n(p_{n+1}) = p_n$, and $\rho_n(q_{n+1}) = q_n$. Thus, we get two coherent sequences of bounded degree, yielding two elements $p, q \in R'$. As in the proof of Proposition 4.6.3, we get that $f = pq$, that is, $f$ is reducible. $\square$

**Corollary 4.6.5.** *If* $f, g \in R'$ *are homogeneous, and if* $g$ *is irreducible, then* $f \not| g$ *implies that* $\rho_n(f) \not| \rho_n(g)$ *for almost all* $n$.

**Proposition 4.6.6.** *The ring* $R'$ *is a unique factorization domain.*

*Proof.* We first note that Lemma 4.6.1 implies that any non-unit element in $R'$ can be written as a finite product of irreducible elements (proof by induction). It remains to show that factorization is unique (up to association with units). To this end, suppose that $b \in R'$ has the two factorizations

$$b = \prod_{i=1}^{r} e_i = \prod_{j=1}^{s} f_j, \quad R' \ni e_i, f_j \text{ irreducible} \tag{4.4}$$

By Lemma 4.6.4, there is an $N$ such that whenever $n > N$, all $\rho_n(e_i)$ and $\rho_n(f_j)$ are irreducible. For such $n$, (4.4) implies that

$$\rho_n(b) = \prod_{i=1}^{r} \rho_n(e_i) = \prod_{j=1}^{s} \rho_n(f_j), \quad K[x_1, \ldots, x_n] \ni \rho_n(e_i), \rho_n(f_j) \text{ irreducible} \tag{4.5}$$

Since $K[x_1, \ldots, x_n]$ is an unique factorization domain, we have immediately that $s = r$. Furthermore, there is a permutation $\sigma \in \Sigma_r$ (the symmetric group on $\{1, \ldots, r\}$) and an $r$-tupel $\mathbf{c} = (c_1, \ldots, c_r) \subset K^r$ such that $f_j = c_j e_{\sigma(j)}$ for $1 \leq j \leq$

$r$. We see that there can be no more than $r!$ such pairs $(\sigma, \mathbf{c})$: each $c_j$ is uniquely determined (by $\sigma$) from the equation $f_j = c_j e_{\sigma(j)}$. Let $\emptyset \neq P(n) \subset \Sigma_r$ denote these permutations. For each pair $(n, \sigma)$, with $n \in \mathbb{N}^+$ and $\sigma \in P(n)$, let $T(n, \sigma)$ denote the uniquely determined $r$-tupel of constants, as above.

For $n' > n$, if $\rho_{n'}(e_i)$ is associated with $\rho_{n'}(f_j)$ then $\rho_n(e_i) = \rho_n(\rho_{n'}(e_i))$ is certainly associated with $\rho_n(f_j) = \rho_n(\rho_{n'}(f_j))$; however, the converse need not hold. At any rate, we have shown that $P(n+1) \subset P(n)$, and demonstrated that $T(n+1, \sigma) = T(n, \sigma)$ whenever $T(n+1, \sigma)$ is defined. Therefore, it suffices to show that there is some $\sigma$ which is in $P(n)$ for all $n$, that is, that $\cap_{n=1}^{\infty} P(n) \neq \emptyset$. Every decreasing sequence of non-empty, finite sets has non-empty intersection, hence the proposition follows.

□

**Example 4.6.7.** Suppose that $e_1 = e_2 = f_1 = f_2 = 1 + x_1 + x_2$, and that $e_3 = f_3 = 1 + x_1$. Then all irreducible factors are irreducible when restricted to $K[x_1]$, that is for $n = 1$, and for all higher $n$. At level $n = 1$ the valid permutations are all of $\Sigma_3$ since all factors restrict to $1 + x_1$ and hence are indistinguishable; for $n > 1$ the valid permutations are the identity permutation and the transposition $(12)$.

**Proposition 4.6.8.** *Let $f, g \in R'$ be homogeneous. Then, for almost all $n$, we have that $\rho_n(\gcd(f, g)) = \gcd(\rho_n(f), \rho_n(g))$.*

*Proof.* First, we deal with the case when $f$ and $g$ are relatively prime, so that $\gcd(f, g) = 1$. We write $f = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $g = \prod_{j=1}^{s} q_j^{\beta_j}$ where the $p_i$'s and $q_j$'s are irreducible. Then no $p_i$ is associated to any $q_j$.

By Lemma 4.6.4 we get that, for almost all $n$, $\rho_n(f) = \prod_{i=1}^{r} \rho_n(p_i)^{\alpha_i}$ and $\rho_n(g) = \prod_{j=1}^{s} \rho_n(q_j)^{\beta_j}$ with $\rho_n(p_i)$ and $\rho_n(p_j)$ irreducible for all $i, j$. Now, pick a pair $(i, j)$. We have that $p_i \not| q_j$, therefore, by Corollary 4.6.5, for almost all $n$, we have that $\rho_n(p_i) \not| \rho_n(q_j)$. Since $(i, j)$ was an arbitrary pair, and since we may similarly conclude that $\rho_n(q_j) \not| \rho_n(p_i)$, we conclude that for almost all $n$, $\rho_n(f)$ and $\rho_n(g)$ are relatively prime. Hence

$$\gcd(\rho_n(f), \rho_n(g)) = 1.$$

If on the other hand $\gcd(f, g) = h \notin K$, then we write $f = f'h$, $g = g'h$, where $f'$ and $g'$ are relatively prime. By the above, for almost all $n$,

$$\begin{aligned}
\rho_n(\gcd(f, g)) &= \rho_n(h \gcd(f', g')) \\
&= \rho_n(h)\rho_n(\gcd(f', g')) \\
&= \rho_n(h) \gcd(\rho_n(f'), \rho_n(g')) \\
&= \gcd(\rho_n(h)\rho_n(f'), \rho_n(h)\rho_n(g')) \\
&= \gcd(\rho_n(hf'), \rho_n(hg')) \\
&= \gcd(\rho_n(f), \rho_n(g)).
\end{aligned}$$

□

**Corollary 4.6.9.** *For* $f, g \in R'$ *homogeneous, and for almost all positive integers* $n$*, we have that*

$$|\gcd(\rho_n(f), \rho_n(g)| = |\gcd(f, g)|.$$

*Proof.* Combine Proposition 4.6.8 and Lemma 4.6.2. □

*Remark 4.6.10.* Clearly, the same result holds for least common multipliers, and for gcd's and lcm's of finite tuples.

## 4.7 Acknowledgements

Johan Andersson provided a short and instructive proof of Lemma 4.4.7. Kristina Crona suggested the present disposition of the material in Section 2, and pointed out some gaps in the argumentation in the first draft of the article. Jörgen Backelin and Ralf Fröberg corrected a misunderstanding of the author regarding the possible Hilbert Series of graded quotient algebras of polynomial algebras, provided the counter-example in Example 4.5.5, and brought to his attention the rolê played by the lattice of principal ideals.

# 5. GRÖBNER BASES FOR NON-HOMOGENEOUS IDEALS IN $R^{'}$

*Abstract*

We extend the Gröbner basis theory developed in [75] to certain non-homogeneous ideals in the ring $R^{'}$, and to certain admissible orders. The main tool used is the study of a homogeneous ideals that may be associated to a non-homogeneous ideal $I \subset R^{'}$, namely the ideal $\mathrm{gr}_{\mathcal{T}}(I)$ generated by all homogenous components of maximal degree of elements in I.

## *5.1 Introduction*

In [75] a Gröbner basis theory for the ring $R^{'}$ is developed. To ensure the existence of normal forms, the above articles consider only homogeneous ideals, which furthermore are required to be locally finitely generated, that is, they have a generating set which contains only finitely many elements of a given total degree.

We call non-homogeneous ideals that fulfill the same property locally filtered finitely generated. We show that any countably generated ideal in $R^{'}$ is locally filtered finitely generated.

To each locally filtered finitely generated ideal I, we may associate a homogeneous ideal, namely the associated graded ideal with respect to the total degree filtration. When this homogeneous ideal is locally finitely generated, we can, if it is explicitly given, calculate its Gröbner basis, and use it to derive a Gröbner basis of the original ideal. We show that this "associated homogeneous ideal" $\mathrm{gr}_{\mathcal{T}}(I)$ is locally finitely generated iff for all d,

$$\dim_K \left( \frac{\mathcal{T}^{\leq d} I}{\sum_{j=1}^{d-1} \mathcal{T}^{\leq j} R' \mathcal{T}^{\leq d-j} I} \right) < \infty.$$

This generalizes the corresponding result for homogeneous ideals.

This article depends heavily on [75], to which we refer the reader.

## 5.2 Preliminaries

Let $K$ be any field, and let $R = K[[x_1, x_2, x_3, \ldots]]$ be the power series ring on countably many variables, with coefficients in a field $K$. Let $R'$ be the smallest sub-algebra of $R$ that contains all homogeneous elements. Let $>$ be an admissible order on the monoid $\mathcal{M}$ of monomials[1] in $R'$, that is, $>$ is a total order that makes $(\mathcal{M}, >)$ into an ordered monoid; furthermore we demand that 1 is the smallest element, and that $x_1 > x_2 > x_3 > \cdots$. By [75, Theorem 5.12], every subset of $\mathcal{M}$ such that the sum of its elements is an element in $R'$ has a maximal element with respect to $>$. We can thus define the *leading power product* $\mathrm{Lpp}(f) \in \mathcal{M}$ for any $f \in R'$ as the maximal element of the set $\mathrm{Mon}(f)$ of the power products that occur in $f$, and also associate to an ideal $I \subset R'$ the monomial ideal $\mathrm{gr}(I)$ that is generated by all leading power products of elements in $I$. It is proved in [75] that if $I$ is locally finitely generated, that is, $I$ is homogeneous and have a generating set that contains only finitely many elements of a given total degree, then so is $\mathrm{gr}(I)$.

There is a natural filtration by total degree on $R'$:

$$\mathcal{T}^{\leq d}R' = \left\{ f \in R' \middle| |f| \leq d \right\}$$

where $|f|$ denotes the total degree of $f$ (by the very definition of $R'$, this is a finite number). One may restrict this filtration to a filtration on any ideal $I$:

$$\mathcal{T}^{\leq 0}I \subset \mathcal{T}^{\leq 1}I \subset \mathcal{T}^{\leq 2}I \subset \mathcal{T}^{\leq 3}I \subset \cdots$$

We shall use the notations $I_{\leq d}$ and $R'_{\leq d}$ as synonyms of $\mathcal{T}^{\leq d}I$ and $\mathcal{T}^{\leq d}R'$. On occasion, we shall write $\mathcal{T}^{<d}$ or $I_{<d}$ for $I_{\leq d-1}$, and so on.

## 5.3 Normal forms with respect to locally filtered finite sets

### 5.3.1 Definition of locally filtered finite sets

In [75], the concept of polynomial normal forms of elements in $R'$, with respect to a finite set, was defined. The definition was then extended to a locally finite set, that is, a set of homogeneous elements such that, for each total degree, only finitely many elements of said degree is contained in the set under consideration. This concept generalizes to sets of non-homogeneous elements:

[1] We mean by a *monomial* a power product $m = \mathbf{x}^{\alpha} = \prod x_i^{\alpha_i}$ where $\alpha_i = 0$ for almost all $i$. Thus, a monomial is always monic. An element in $R'$ of the form $cm$, with $c \in K$ and $m$ a monomial, is called a *term*.

**Definition 5.3.1.** A subset $F$ of $R'$ is said to be *locally filtered finite* if, for each total degree $d$, it contains only finitely many elements of total degree $d$.

An ideal $I$ in $R'$ is said to be *locally filtered finitely generated* if it is generated by a locally filtered finite set.

**Lemma 5.3.2.** *An ideal $I$ of $R'$ is locally filtered finitely generated iff it is countably generated.*

*Proof.* A locally filtered finite set is obviously countable, hence a locally filtered finitely generated ideal is countably generated.

To prove the converse, let $F = \{f_i\}_{i=1}^{\infty}$ be a countable generating set for $I$. Put $d(j) = \max\{|f(i)||i \leq j\}$. Now, we define $g_1 = f_1$ and inductively $g_j = f_j + f_1^{d(j)+j}$. The set $G = \{g_i\}_{i=1}^{\infty}$ is easily seen to be locally filtered finite. In fact, for each total degree $t$, there can be at most one element in $G$ with total degree $t$. It is obvious that $G$ generates $I$. □

**Example 5.3.3.** There exists non-countably generated ideals in $R'$. Consider, for instance, the graded maximal ideal $R'^{+} = \coprod_{i=1}^{\infty} R'_i$ consisting of all power series in $R'$ with zero constant term.

### *5.3.2 Degree-compatible reduction systems*

**Lemma 5.3.4.** *Let $F = \{f_1, \ldots, f_r\} \subset R'$ consist of monic, homogeneous elements. Let $h \in R'$ be homogeneous. Then the polynomial normal forms of $h$ consist of homogeneous elements with total degree $|h|$ (and possibly the zero element). If on the other hand $h$ and the elements of $F$ are not necessarily homogeneous, but the elements of $F$ have the property that*

$$(f \in F) \wedge (m, m' \in \mathrm{Mon}(f)) \wedge (|m| > |m'|) \implies m > m'$$

*then we have that the normal forms of $F$ have total degree $\leq |h|$. In particular, this holds if $>$ takes total degree first, that is, if*

$$(m, m' \in \mathcal{M}) \wedge (|m| > |m'|) \implies m > m'$$

*Proof.* (Sketch) Recall from [75] that the normal forms of $h$ are formed by choosing an integer $n$ large enough so that $1 \leq i \leq r$, $j \geq n$ implies that $x_j \not| \mathrm{Lpp}(f_i)$, and then regarding $R'$ as a subring of

$$K[[x_{n+1}, x_{n+2}, x_{n+3}, \ldots]][x_1, \ldots, x_n].$$

This was used to demonstrate that the normal form of $h$ is obtained by a finite number of substitutions $\mathrm{Lpp}(f_i) \mapsto (f_i - \mathrm{Lpp}(f_i))$. If all $f_i$ are homogeneous, then each substitution preserves the total degree; if the other condition is fulfilled, then $|\mathrm{Lpp}(f_i)| \geq |f_i - \mathrm{Lpp}(f_i)|$, so that each substitution either preserves the total degree, or lowers it. □

Inspired by the above, we make the following definitions:

**Definition 5.3.5.** We call a pair $(F, >)$, $F$ a subset of $R'$, $<$ an admissible order on $\mathcal{M}$, a *reduction system*; we will often show it as

$$\begin{bmatrix} \mathrm{Lpp}(f_1) & \mapsto & f_1 - \mathrm{Lpp}(f_1) \\ \mathrm{Lpp}(f_2) & \mapsto & f_2 - \mathrm{Lpp}(f_2) \\ \mathrm{Lpp}(f_3) & \mapsto & f_3 - \mathrm{Lpp}(f_3) \\ & \vdots & \end{bmatrix} \tag{5.1}$$

where the $f_i$'s are the elements[2] of $F$, and the leading power product is defined by means of $>$.

**Definition 5.3.6.** A reduction system $(F, >)$ is called *degree-compatible* if

$$(f \in F) \wedge (m, m' \in \mathrm{Mon}(f)) \wedge (|m| > |m'|) \implies m > m'$$

An element $f \in R'$ is called degree-compatible (with respect to $>$) if $(\{f\}, >)$ is degree-compatible. An admissible order $>$ is said to be degree-compatibleif $(F, >)$ is degree-compatible for all subsets $F$ of $R'$.

**Lemma 5.3.7.** $(F, >)$ *is degree-compatible iff each* $f \in F$ *is degree-compatible (with respect to* $>$*).*

*If* $(F, >)$ *is degree-compatible, and* $f \in R'$ *is degree-compatible (with respect to* $>$*), then any normal form of* $f$ *(with respect to the reduction system* $(F, >)$*) is degree-compatible (with respect to* $>$*).*

*An admissible order* $>$ *is degree-compatible iff* $>$ *coincides with* $>_{\mathrm{tot}}$ *on* $\mathcal{M}$*. Here,* $>_{\mathrm{tot}}$ *denotes the degree-compatible order obtained from* $>$ *by* $m >_{\mathrm{tot}} m'$ *if* $|m| > |m'|$ *or if* $|m| = |m'|$ *and* $m > m'$*.*

**Example 5.3.8.** If $(F, >)$ is degree-compatible, and $f \in R'$ is *not* degree-compatible, then a normal form of $f$ need not be degree-compatible. To see this, consider the reduction system $(\{x_1\}, >_{\mathrm{lex}})$ and the element $x_2 - x_3^3$, which is in normal form.

**Example 5.3.9.** If $(F, >)$ is degree-compatible, then an element of the ideal (or indeed sub-algebra) generated by $F$ need not be degree-compatible (with respect to $>$). Consider for instance $F = \{x_1, x_2^2 + x_2\}$ and let $>$ be the (pure) lexicographic order. Then $(F, >)$ is degree-compatible, whereas

$$\left(\{x_1 + x_2^2 + x_2\}, >\right)$$

is not.

[2] If $F$ is finite or countable, which it shall be for our applications, the enumeration of the elements of $F$ is straightforward. Should the need arise to consider larger sets, we can appeal to the *Well-Ordering Theorem* (see [49]) to get a well-ordered index set (for $F$) which contains the positive natural numbers as a proper initial segment. This motivates the pictorial description (5.1).

**Example 5.3.10.** If $(F, >) = (\{x_1 - x_2^2 - x_3^3\}, >)$ where $>$ denotes the pure lexicographic order, then $x_1 \in R'$ is degree-compatible whereas its normal form $x_2^2 - x_3^3$ is not.

### 5.3.3 Normal forms with respect to degree-compatible reduction systems

In our new vocabulary, we can formulate Lemma 5.3.4 as follows:

**Corollary 5.3.11.** *The polynomial normal forms of an element $f \in R'$, with respect to a (finite) degree-compatible reduction systems have total degree $\leq |f|$.*

Hence, the observation in [75], that when reducing an element of total degree $t$ with respect to a (homogeneous) locally finitely generated set, we need only consider the finite subset of elements with total degree $\leq t$, is valid also for this situation. It follows that there always exists polynomial normal forms with respect to such a set. We conclude:

**Theorem 5.3.12 (Division algorithm for locally filtered finite sets).** *Let $F$ be a locally filtered finite subset of $R'$, $>$ be an admissible order such that $(F, >)$ is degree-compatible, $h$ be an element in $R'$. Then there exists an admissible combination*[3] *$L$ of elements in $F$, and a remainder term $Q$ (called a normal form of $h$), such that*

*(i)* $h = L + Q$

*(ii) If* $\mathrm{Mon}(h) \cap \langle \mathrm{in}(F) \rangle = \emptyset$*, then* $L = 0$ *and* $Q = h$.

*(iii) Otherwise,* $L \neq 0$ *and either* $Q = 0$ *or* $\mathrm{Mon}(Q) \cap \langle \mathrm{in}(F) \rangle = \emptyset$.

**Example 5.3.13.** If $(F, >)$ is not degree-compatible, then things may go astray. Let $>_{lex}$ denote the (pure) lexicographic order on $\mathcal{M}$, and let

$$F = \{x_1 - x_2^2, x_2^2 - x_3^3, x_3^3 - x_4^4, \ldots, x_n^n - x_{n+1}^{n+1}, \ldots\}.$$

Then $F$ is locally filtered finite. Now, the resulting reduction system, with respect to $F$ and $>_{lex}$, is

$$\begin{bmatrix} x_1 & \mapsto & x_2^2 \\ x_2^2 & \mapsto & x_3^3 \\ x_3^3 & \mapsto & x_4^4 \\ & \vdots & \\ x_n^n & \mapsto & x_{n+1}^{n+1} \\ & \vdots & \end{bmatrix} \tag{5.2}$$

[3] We adopt the convention that 0 is an admissible combination of zero elements.

It is clear that $x_1$ has no normal form with respect to (5.2).

On the other hand, if we use the total degree, then lexicographic order, we get the reduction system

$$\begin{bmatrix} x_2^2 \mapsto x_1 \\ x_3^3 \mapsto x_2^2 \\ x_4^4 \mapsto x_3^3 \\ \vdots \\ x_n^n \mapsto x_{n-1}^{n-1} \\ \vdots \end{bmatrix} \quad \text{or equivalently} \quad \begin{bmatrix} x_2^2 \mapsto x_1 \\ x_3^3 \mapsto x_1 \\ x_4^4 \mapsto x_1 \\ \vdots \\ x_n^n \mapsto x_1 \\ \vdots \end{bmatrix} \tag{5.3}$$

As stated in Theorem 5.3.12, every element has a normal form with respect to (5.3); in particular, $x_1$ is already in normal form. A general monomial $\prod_{i=1}^{N} x_i^{\alpha_i}$ have normal form $\prod_{i=1}^{N} x_i^{\beta_i}$ where for $i > 1$ we have that $\beta_i = \mathrm{Rem}(\alpha_i, i)$; the coefficient $\beta_1$ is equal to $\alpha_1 + \sum_{i=2}^{N} \left\lfloor \frac{\alpha_i}{i} \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part, and Rem the remainder.

### 5.3.4 Normal forms with respect to ideals in $R'$

If $I$ is a homogeneous, locally finitely generated ideal in $R'$, then we may calculate a locally finite Gröbner basis $F$ for $I$, with respect to an arbitrary admissible order $>$ [75]: that means, that the set $\{\mathrm{Lpp}(f) | f \in F\}$ generates the initial ideal $\mathrm{gr}(I) = \{\mathrm{Lpp}(g) | g \in I\}$. This $F$, together with $>$, constitute a degree-compatible reduction system, with the extra property that each element of $R'$ have a unique normal form with respect to the reduction system. Hence, we may view the calculation of normal form as a map $N : R' \to R'$ with the property that $N(I) = \{0\}$ and $N \circ N = N$. It is clear that (as long as $F$ is a Gröbner basis for $I$ w.r.t $>$) this map only depends on $I$ and $>$, and not on the choice of $F$.

We would similarly like to be able to calculate normal forms with respect to non-homogeneous ideals. However, as the following example shows, we can not hope to do so for arbitrary locally filtered finitely generated ideals.

**Example 5.3.14.** By Lemma 5.3.2, any countably generated ideal is locally filtered finitely generated. Hence, the ideal $x_1, x_2, x_3, x_4, \ldots$ is in fact locally filtered finitely generated.[4] Consider the element $f = x_1 + x_2 + x_3 + \cdots \in R'$. It is clear that $f$ can have no normal form with respect to $I$.

On the other hand, if a non-homogeneous ideal possesses a locally filtered finite Gröbner basis $F$, with respect to a degree-compatible admissible order $>$,

[4] A slight modification of the recipe outlined in the proof of Lemma 5.3.2 yields the locally filtered finite generating set $x_1, x_1^2 + x_2, x_1^3 + x_3, x_1^4 + x_4, \ldots$.

then, clearly, the pair $(F, >)$ constitute a degree-compatible reduction system. As in the homogeneous case, we get a uniquely determined normal form map:

**Lemma 5.3.15.** *Let $I$ be a countably generated ideal in $R'$, and suppose that $F$ is a locally filtered finite Gröbner basis for $I$, with respect to a degree-compatible admissible order $>$. Then, $(F, >)$ is a degree-compatible reduction system. Each element in $R'$ has a unique normal form with respect to this reduction system.*

*Proof.* It is immediate that $(F, >)$ is a degree-compatible reduction system. Since $F$ is locally filtered finite, Theorem 5.3.12 shows that each element $g \in R'$ has at least one normal form with respect to $(F, >)$. Since $\{\mathrm{Lpp}(f) | f \in F\}$ generates $\mathrm{gr}(I)$, it follows by standard arguments, using Theorem 5.3.12, that normal forms are unique. $\square$

## 5.4 Graded associated ideals

### 5.4.1 The total-degree filtration

**Definition 5.4.1.** The graded associated ideal (with respect to the total-degree filtration) is defined by

$$\mathop{\mathrm{gr}}_{\mathcal{T}}(I) = \bigoplus_{t \in \mathbb{N}} \frac{\mathcal{T}^{\leq t} I}{\mathcal{T}^{< t} I}.$$

This can be regarded as the homogeneous ideal in $R'$ that is generated by all homogeneous components of maximal degree of elements in $I$. Thus, if we define $c(f)$ as the highest homogeneous component of $f$, then

$$\mathop{\mathrm{gr}}_{\mathcal{T}}(I) \simeq \{c(f) | f \in I\}.$$

**Lemma 5.4.2.** *If $\mathrm{gr}_{\mathcal{T}}(I)$ is locally finitely generated then $I$ is locally filtered finitely generated.*

*Proof.* Let $F$ be a locally finite generating set for $\mathrm{gr}_{\mathcal{T}}(I)$. Then, for each $f \in F$ we can find a $\tilde{f} \in I$ such that $f$ is the homogeneous component of maximal degree of $\tilde{f}$. Denote by $\tilde{F}$ the set of all $\tilde{f}$. We claim that $\tilde{F}$ is a locally filtered finite generating set for $I$.

That $\tilde{F}$ is locally filtered finite is immediate. To see that it generates $I$, choose $h \in I$ and write it as a sum of homogeneous components, $h = \sum_{i=0}^{r} h_i$. Clearly, $c(h) = h_r$ is an element of $\mathrm{gr}_{\mathcal{T}}(I)$ and can be generated by elements in $F$, $h_r = \sum_{k=1}^{t} g_k f_k$. Now, the corresponding expression $h - \sum_{k=1}^{t} g_k \tilde{f}_k \in I$ need not be zero, but it will have total degree $< r$. By induction, the result follows. $\square$

*Remark 5.4.3.* The converse of Lemma 5.4.2 does not hold. Consider any homogeneous, countably generated, non-locally finitely generated ideal, such as for instance $(x_1, x_2, x_3, x_4, \dots)$. Then, by Lemma 5.3.2, this ideal is locally filtered finitely generated but not locally finitely generated. Furthermore, homogeneous ideals coincide with their graded associated ideal (with respect to the total-degree filtration).

**Lemma 5.4.4.** *If $I$ is an locally filtered finitely generated ideal, and $F \subset I$ is such that the set $\{c(f) | f \in F\}$ generates $\mathrm{gr}_{\mathcal{T}}(I)$, then $F$ generates $I$.*

*If $\{c(f) | f \in F\}$ generates $\mathrm{gr}_{\mathcal{T}}(I)$ up to degree $d$, then $F$ generates $I$ up to degree $d$.*

*Proof.* This is a straightforward modification for the corresponding results for ordinary Gröbner bases, and for so-called homogeneous (Macaulay) bases. □

**Proposition 5.4.5.** *If $I$ is an ideal in $R'$, and if*

$$\forall k > d : \frac{I_{\le k}}{\sum_{j=1}^{k} R'_{\le j} I_{\le k-j}} = 0,$$

*then $I$ is generated in degrees $\le d$, that is, $\langle I_{\le d} \rangle_{R'} = I$.*

*Proof.* The condition is equivalent to

$$\forall k > d : \qquad I_{\le k} = \sum_{j=1}^{k} R'_{\le j} I_{\le k-j} \tag{5.4}$$

We must prove that $I_{\le k} = (\langle I_{\le d} \rangle_{R'})_{\le k}$. Taking $k = d+1$ in (5.4), we get that

$$I_{\le d+1} = \sum_{j=1}^{d+1} R'_{\le j} I_{\le d+1-j} = \sum_{\nu=0}^{d} R'_{\le d+1-\nu} I_{\le \nu} \subset \langle I_{\le d} \rangle_{R'} \tag{5.5}$$

Proceeding by induction, we get that $I_{\le d+r} \subset \langle I_{\le d} \rangle_{R'}$ for all $r \ge 0$. □

The following, quite general, theorem will have interesting applications to ideals in the filtered ring $R'$:

**Theorem 5.4.6.** *Let $T = \cup_{i=0}^{\infty} T_{\le i}$ be a (commutative) filtered ring, and let $I = \cup_{i=0}^{\infty} I_{\le i}$ be a filtered $T$-module. Denote by $S$ the graded associated ring to $T$, and by $J$ the graded associated $S$-module to $I$. Then for all integers $d \ge 1$, ($d \ge 0$ if we adopt the convention that an empty sum corresponds to the zero group)*

$$\frac{I_{\le d}}{\sum_{j=1}^{d} T_{\le j} I_{\le d-j}} \simeq \frac{J_d}{\sum_{j=1}^{d} S_j J_{d-j}} \tag{5.6}$$

*as abelian groups. If in addition $T_{\le i}$ and $I_{\le i}$ are $K$-vector spaces for all non-negative integers, then (5.6) is an isomorphism of $K$-vector spaces.*

*Proof.* An elementary chase in the diagram

$$\begin{tikzcd}
\mathrm{I}_{\leq d} \arrow[r, two heads, "\alpha"] \arrow[d, two heads, "\beta"'] & \frac{\mathrm{I}_{\leq d}}{\sum_{j=1}^{d-1} \mathrm{T}_{\leq j} \mathrm{I}_{\leq d-j}} \arrow[ddl, "\varphi"] \\
\mathrm{J}_d \arrow[d, two heads, "\gamma"'] & \\
\frac{\mathrm{J}_d}{\sum_{j=1}^{d-1} \mathrm{S}_j \mathrm{J}_{d-j}} \arrow[uur, "\psi"] &
\end{tikzcd} \tag{5.7}$$

where $\alpha, \beta, \gamma$ are the natural quotient maps, and $\varphi, \psi$ are defined by lifts and compositions, shows that $\varphi$ and $\psi$ are mutual inverses. Note that if $h \in I_{\leq r+s}$, with $h = af$, $a \in T_{\leq r}$, $f \in I_{\leq s}$, then $\beta(h) = \epsilon(a)\delta(f)$, where $\epsilon : T_{\leq r} \twoheadrightarrow S_r$ and $\delta : I_{\leq s} \twoheadrightarrow I_{\leq s}/I_{\leq s-1} \simeq J_s$ are the natural quotient epimorphisms. $\square$

**Theorem 5.4.7.** *For a proper locally filtered finitely generated ideal* $I$ *in* $R'$*, the following are equivalent:*

*(i)* $\mathrm{gr}_{\mathcal{T}}(I)$ *is locally finitely generated,*

*(ii)* $\forall d : \dim_K \left( \frac{I_{\leq d}}{\sum_{j=1}^{d-1} R'_{\leq j} I_{\leq d-j}} \right) < \infty.$

*(iii)* $\forall d : \dim_K \left( \frac{\mathrm{gr}_{\mathcal{T}}(I)_d}{\sum_{j=1}^{d-1} R'_j \, \mathrm{gr}_{\mathcal{T}}(I)_{d-j}} \right) < \infty.$

*Proof.* The equivalence $(i) \iff (iii)$ follows from the fact [75] that a (proper) homogeneous ideal $J$ is locally finitely generated iff

$$\dim_K \left( \frac{J_g}{\sum_{j=1}^{g-1} R'_j J_{g-j}} \right) < \infty$$

for all $g$. The present theorem is a generalization of this result to non-homogeneous ideals.

The equivalence $(ii) \iff (iii)$ follows from Theorem 5.4.6, applied to the filtered (by total degree) ring $R'$ and the filtered module $I$. $\square$

**Proposition 5.4.8.** *The converse of Proposition 5.4.5 does not hold, even for finitely generated ideals. That is, there exists a finitely generated ideal* $I$ *in* $R'$*, generated in degrees* $\leq d$*, for which*

$$\frac{I_{\leq k}}{\sum_{j=1}^{k-1} R'_{\leq j} I_{\leq k-j}} \neq 0$$

*for some* $k > d$*.*

*Proof.* Let $I$ be any finitely generated (non-homogeneous) ideal generated in degrees $\leq d$ for which the associated homogeneous ideal $\mathrm{gr}_{\mathcal{T}}(I)$ is *not* generated in degrees $\leq d$. Such ideals exist already in the polynomial rings $K[x_1, \dots, x_n]$; we may extend such an ideal to $R'$ via the inclusion, and get an example. For instance, if we put $f = xy^2 + y^3 + x^2$, $g = x^2y$, then $x^4 = x^2f - (xy + y^2)g$ can not be written as a combination of $c(f) = xy^2 + y^3$ and $c(g) = x^2y$, so if $I = (f, g)$ then $I$ is generated in degrees $\leq 3$ but $\mathrm{gr}_{\mathcal{T}} I$ has minimal generators of degree 4.

Suppose, towards a contradiction, that the converse of Proposition 5.4.5 does in fact hold. Then

$$\frac{I_{\leq k}}{\sum_{j=1}^{k-1} R'_{\leq j} I_{\leq k-j}} = 0$$

for all $k > d$. By Theorem 5.4.6, this implies that

$$\frac{\mathrm{gr}_{\mathcal{T}}(I)_k}{\sum_{j=1}^{k-1} R'_j \, \mathrm{gr}_{\mathcal{T}}(I)_{k-j}} = 0$$

for all $k > d$. But for the homogeneous ideal $\mathrm{gr}_{\mathcal{T}}(I)$, it is clear that this is equivalent to that $\mathrm{gr}_{\mathcal{T}}(I)$ is generated in degrees $d$. We have assumed that this is not the case, a contradiction. □

**Question 5.4.9.** *If $I$ is a finitely generated ideal in $R'$, is the associated homogeneous ideal then finitely generated, or at least locally finitely generated?*

### 5.4.2 The termorder filtration

**Definition 5.4.10.** If $>$ is an admissible order on $\mathcal{M}$, then denote by $\mathcal{F}^{\leq m}R'$ the set of elements with leading power product $\leq m$. This restricts to a filtration on any ideal $I$.

We note that the initial ideal $\mathrm{gr}(I)$ of an ideal $I$ of $R'$ can be thought of as the graded associated object associated to the filtration $\mathcal{F}$. If $>$ is degree-compatible, then $\mathcal{F}$ is a refinement of the total-degree filtration $\mathcal{T}$.

This situation merits a closer study. We have two filtrations on $I$, and the graded associated ideal for each filtration is realizable as an ideal in $R'$. If the operation of forming graded associated objects is associative, we have that the initial ideal of the associated homogeneous ideal to $I$ equals the initial ideal of $I$ itself. As it happens, this is in fact true.

To make precise what we mean by saying that one filtration is a refinement of another, we observe that (in the cases that we are interested in) a filtration $\mathcal{G}$ (of an abelian group $V$, say) indexed by a totally ordered set $P$ (in most cases, an ordered semigroup such as $\mathbb{N}$ or $\mathcal{M}$) is determined by the associated valuation $\phi : V \to P$ that maps each element $v \in V$ to its filtration degree $p \in P$, which is the smallest $p$ such that $v \in \mathcal{G}^p$. This is a surjective map, assuming that the filtration is exhaustive, and the filtration subgroups are given by inverse images $\phi^{-1}([0,p))$ or $\phi^{-1}([0,p])$. If $\xi : V \to Q$ is another filtration of $V$, $P$ is said to be an refinement of $Q$ if there is an order-preserving surjection $\pi$ from $P$ to $Q$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, two heads, "\phi"] \arrow[dr, "\xi"'] & P \arrow[d, two heads, "\pi"] \\ & Q \end{tikzcd} \tag{5.8}$$

The fact that "taking graded associated is an associative operation" is asserted in the rather technical lemma below.

**Lemma 5.4.11.** *Let $V$ be an abelian group, and let $P, Q$ be two totally ordered sets with a minimal element, which we denote by $0$. Assume that there are given surjective maps (taking $0$ to $0$) $\phi : V \to P$ and $\xi : V \to Q$, and an order-preserving surjection $\pi : P \to Q$ such that $\xi = \pi \circ \phi$. Define $A^{\leq p}V = \phi^{-1}([0,p])$, $A^{<p}V = \phi^{-1}([0,p))$, $B^{\leq q}V = \xi^{-1}([0,q])$, $B^{<q}V = \xi^{-1}([0,q))$,*

$$\operatorname*{gr}_A(V) = \bigoplus_{p\in P} \frac{A^{\leq p}V}{A^{<p}V},$$

*and*

$$\operatorname*{gr}_B(V) = \bigoplus_{q\in Q} \frac{B^{\leq q}V}{B^{<q}V}.$$

*Give $\mathrm{gr}_B(V)$ an induced $A$-filtration by*

$$A^{\leq p}\frac{B^{\leq q}V}{B^{<q}V} = \begin{cases} 0 & \textit{if } \pi(p) < q \\ \frac{B^{\leq q}V}{B^{<q}V} & \textit{if } \pi(p) > q \\ \frac{A^{\leq p}V}{B^{<q}V} & \textit{if } \pi(p) = q \end{cases}$$

$$A^{<p}\frac{B^{\leq q}V}{B^{<q}V} = \begin{cases} 0 & \textit{if } \pi(p) < q \\ \frac{B^{\leq q}V}{B^{<q}V} & \textit{if } \pi(p) > q \\ \frac{A^{<p}V}{B^{<q}V} & \textit{if } \pi(p) = q \end{cases}$$

*Define*

$$\operatorname*{gr}_A(\operatorname*{gr}_B(V)) = \bigoplus_{p\in P} \frac{A^{\leq p}\,\mathrm{gr}_B(V)}{A^{<p}\,\mathrm{gr}_B(V)} = \bigoplus_{p\in P}\bigoplus_{q\in Q} \frac{A^{\leq p}\frac{B^{\leq q}V}{B^{<q}V}}{A^{<p}\frac{B^{\leq q}V}{B^{<q}V}}.$$

*Then $\mathrm{gr}_A(V)$ and $\mathrm{gr}_A(\mathrm{gr}_B(V))$ are isomorphic as $P$-graded abelian groups.*

*Proof.* From the definition, we have that the only non-zero terms of

$$\operatorname*{gr}_A(\operatorname*{gr}_B(V)) = \bigoplus_{p\in P}\bigoplus_{q\in Q} \frac{A^{\le p}\frac{B^{\le q}V}{B^{<q}V}}{A^{<p}\frac{B^{\le q}V}{B^{<q}V}}$$

are those where $\pi(p) = q$, thus

$$\operatorname*{gr}_A(\operatorname*{gr}_B(V)) = \bigoplus_{p\in P}\bigoplus_{q\in Q} \frac{A^{\le p}\frac{B^{\le q}V}{B^{<q}V}}{A^{<p}\frac{B^{\le q}V}{B^{<q}V}} = \bigoplus_{p\in P} \frac{\frac{A^{\le p}V}{B^{<\pi(p)}V}}{\frac{A^{<p}V}{B^{<\pi(q)}V}} = \bigoplus_{p\in P} \frac{A^{\le p}V}{A^{<p}V} = \operatorname*{gr}_A(V),$$

by application of the Third Isomorphism Theorem. □

**Lemma 5.4.12.** *If* $I$ *is any ideal of* $R'$*, and if* $>$ *is degree-compatible, then*

$$\operatorname{gr}(I) = \operatorname*{gr}_{\mathcal{F}}(I) = \operatorname*{gr}_{\mathcal{F}}(\operatorname*{gr}_{\mathcal{T}}(I)).$$

*Proof.* Take $V = I$, $P = \mathcal{M}$, $Q = \mathbb{N}$, $\phi = \mathrm{Lpp} : I \to \mathcal{M}$, $\xi, = |\cdot| : I \to \mathbb{N}$ and $\pi = |\cdot| : \mathcal{M} \to \mathbb{N}$; then use Lemma 5.4.11. □

## 5.5 Gröbner bases for locally filtered finitely generated ideals in $R'$

**Theorem 5.5.1.** *If* $I$ *is an ideal of* $R'$ *and if* $>$ *is a degree-compatible admissible order, then* $\operatorname{gr}_{\mathcal{T}}(I)$ *is locally finitely generated iff* $\operatorname{gr}(I)$ *is locally finitely generated.*

*Proof.* Assume that $\operatorname{gr}_{\mathcal{T}}(I)$ is locally finitely generated. Since $>$ is degree-compatible, we get by applying Lemma 5.4.12 that $\operatorname{gr}(I) = \operatorname{gr}_{\mathcal{F}}(\operatorname{gr}_{\mathcal{T}}(I))$. Hence, $\operatorname{gr}(I)$ is the initial ideal of the homogeneous, locally finitely generated ideal $\operatorname{gr}_{\mathcal{T}}(I)$. By [75, Theorem 4.8] this is a locally finitely generated ideal.

For the converse, we use Lemma 5.4.12 again to reduce to the case when $I$ is homogeneous. Then, we use the fact that a locally finite Gröbner basis for $I$ is also a locally finite generating set. □

**Theorem 5.5.2.** *Suppose that* $I$ *is an ideal of* $R'$*, that* $>$ *is degree-compatible, and that* $\operatorname{gr}_{\mathcal{T}}(I)$ *is locally finitely generated. If* $G$ *is a locally finite (and homogeneous) Gröbner basis for* $\operatorname{gr}_{\mathcal{T}}(I)$*, then any "lift"* $V$ *of* $G$ *is a locally filtered finite Gröbner basis for* $I$*. By a "lift"* $V$ *we mean that we can write* $G = \{g_\alpha | \alpha \in A\}$*,* $V = \{v_\alpha | \alpha \in A\}$ *for some index set* $A$*, and that* $c(v_\alpha) = g_\alpha$ *for all* $\alpha \in A$*.*

*Conversely, if* $V$ *is a locally filtered finite Gröbner basis for* $I$*, then* $G = \{c(v) | v \in V\}$ *is a locally finite Gröbner basis for* $\operatorname{gr}_{\mathcal{T}}(I)$*).*

*Proof.* By the assumptions, the set $\{\mathrm{Lpp}(g) | g \in G\}$ generates $\mathrm{gr}(\mathrm{gr}_{\mathcal{T}}(I)) = \mathrm{gr}(I)$. Since $>$ is degree-compatible, it is clear that $\mathrm{Lpp}(g_\alpha) = \mathrm{Lpp}(v_\alpha)$ for all $\alpha \in A$. Hence, the set of monomials $\{\mathrm{Lpp}(v_\alpha) | \alpha \in A\}$ generates $\mathrm{gr}(I)$. Clearly, $V$ is a locally filtered finite set.

Conversely, if we have that $\{\mathrm{Lpp}(v) | v \in V\}$ generates $\mathrm{gr}(I) = \mathrm{gr}(\mathrm{gr}_{\mathcal{T}}(I))$, it follows from the fact that $>$ is degree-compatible that $\mathrm{Lpp}(v) = \mathrm{Lpp}(c(v))$ for all $v \in V$. Hence, the set of monomials $\{\mathrm{Lpp}(c(v)) | v \in V\}$ generates $\mathrm{gr}(\mathrm{gr}_{\mathcal{T}}(I))$. Clearly, $G$ is a homogeneous, locally finite set. □

**Theorem 5.5.3.** *Let $>$ be a degree-compatible admissible order on $\mathcal{M}$, and let $I$ be an ideal in $R'$ that is generated by a locally filtered finite set $F$. Suppose that $\mathrm{gr}_{\mathcal{T}}(I)$ is locally finitely generated. Then there exists a locally filtered finite superset $H$ of $F$ such that $H$ is a Gröbner basis for $I$.*

*Proof.* We know from [75] that there exists a homogeneous and degree-finite Gröbner basis $G$ of $\mathrm{gr}_{\mathcal{T}}(I)$. By Theorem 5.5.2, there is a "lift" $V$ of $G$ which is a locally filtered finite Gröbner basis of I. Now put $H = V \cup F$. □

We summarize: all countably generated ideals have a locally filtered finite *generating set*, but only those that have a locally finitely generated associated homogeneous ideal have a locally filtered finite *Gröbner basis*.

## 5.6 Acknowledgment

This article has benefited greatly from comments made by Jörgen Backelin and Ralf Fröberg. Many of the examples, and some of the proofs, are due to them. The idea of focusing the property of a non-homogeneous ideal of having a locally finitely generated associated homogeneous ideal, rather than the property of having a locally filtered finite generating set, was conceived during a discussion with the two of them.

# 6. TOPOLOGICAL PROPERTIES OF $R'$


*Abstract*

We study the power series ring $R = K[[x_1, x_2, x_3, \dots]]$ on a countably infinite number of variables over a field $K$, and in particular its subring $R'$ generated by all homogeneous elements in $R$. By means of a certain decreasing filtration of ideals, which are kernels of the "truncation homomorphisms" $\rho_n : R' \to K[x_1, \dots, x_n]$, we endow $R'$ with a topology, and show that with respect to this topology, homogeneous, finitely generated ideals are closed (as are so-called locally finitely generated ideals).


## *6.1 Introduction*

The power series ring (over a field $K$) $R = K[[x_1, x_2, x_3, \dots]]$ on countably many variables has been the topic of many studies [50, 64, 65, 66]. As a contrast, the two subrings $R'$ and $\tilde{R}$, defined below, are seldom seen in the literature, although $\tilde{R}$, or some variants of it, is known in combinatorics as "the ring of formal polynomials" [15]. From the author's point of view, the "purpose" of the ring $R'$, which is defined as the smallest $K$-subalgebra of $R$ that contains all homogeneous elements, is that it allows the definition of *generic forms* in infinitely many variables. The truncation homomorphisms $\rho_n : R' \twoheadrightarrow K[x_1, \dots, x_n]$ are useful for relating these generic forms with ordinary generic forms in polynomial rings over $K$.

It is an interesting fact [75] that so-called Gröbner bases can be calculated for a wide class of homogeneous ideals, the so-called locally finitely generated ideals, containing the finitely generated homogeneous ideals. By a locally finitely generated ideal we mean a homogeneous ideal that can be generated by a (possibly infinite) homogeneous set, containing only finitely many elements of any given total degree. Initial ideals can therefore be calculated for such ideals, and these initial ideals are also locally finitely generated. In particular, the initial ideals of ideals generated by finitely many generic forms (so called *generic ideals*) are locally finitely generated. There are exist many interesting and non-trivial examples of generic ideals that have (lexicographically) initial ideals that are locally finitely generated but not finitely generated. On the other hand, for such generic ideals, the initial ideal with respect to the graded reverse lexicographic order is always finitely generated [77].

In [76], we related these initial ideals to a countable family of initial ideals of "restricted" ideals in ordinary polynomial rings; we showed that, in some sense, they are the limit of said family. For the special case of generic ideals, this means that if we want to study the initial ideals of i.e. the generic ideal generated by a quadratic and a cubic generalized form, we can approximate this ideal by the initial ideals of the corresponding generic ideals of the various polynomial rings.

We study the topology that the filtration given by the kernels of the truncation homomorphisms induce on $R'$. By showing that locally finitely generated ideals are closed, we answer a question of [76], "Are locally finitely generated ideals determined by their truncated ideals?" affirmatively.

This closedness result is used to prove that if $f_1, \ldots, f_r$ are homogeneous elements in $R'$, and if, for all $n$, the lattice generated by the principal ideals $(\rho_n(f_1))$ to $(\rho_n(f_r))$ form a distributive sublattice of the lattice of ideals in $K[x_1, \ldots, x_n]$, then the lattice generated by the principal ideals $(f_1), \ldots, (f_r)$ is a distributive sublattice of the modular lattice of ideals in $R'$. It remains an open question if, conversely, the distributivity of the lattice generated by $(f_1), \ldots, (f_r)$ implies the distributivity of the "truncated" lattices, for almost all $n$.

## 6.2 Preliminaries

Let $K$ be a field. Denote by $R = K[[x_1, x_2, x_3, \ldots]]$ the power series ring on countably infinitely many variables over $K$. We can provide this ring with an increasing filtration of subgroups

$$R_{\leq -1} = 0 \subset R_0 = K \subset R_{\leq 1} \subset R_{\leq 2} \subset R_{\leq 3} \subset \cdots \tag{6.1}$$

where for $d \in \mathbb{N}$, $R_{\leq d}$ denotes the set of elements of total degree $\leq d$. The filtration (6.1) is not exhaustive, since there exists elements in $R$ of unbounded total degree (such as $\sum_{i=1}^{\infty} x_1^i$), but it is separated. The graded $K$-algebra $R' := \cup_{d=0}^{\infty} R_{\leq d}$ is the smallest $K$-subalgebra of $R$ that contains all homogeneous elements. This ring will be our main object of study.

For any positive integer $n$, the power series ring $K[[x_1, \ldots, x_n]]$ is both a subalgebra and a quotient of $R$, since $R/B_n \simeq K[[x_1, \ldots, x_n]]$, where $B_n$ is the $R$-ideal generated by all power series in $K[[x_{n+1}, x_{n+2}, \ldots]]$ with zero constant term. Therefore, we can define a $K$-algebra epimorphism $\rho_n$, called the $n$'th *truncation homomorphism*, by means of the composite

$$R \twoheadrightarrow R/B_n \simeq K[[x_1, \ldots, x_n]]. \tag{6.2}$$

Let $\mathcal{M}$ be the free commutative monoid on $\{x_1, x_2, x_3, \ldots\}$, and write $\mathcal{M}^n$ for the free commutative submonoid of $\mathcal{M}$ that is generated by $x_1, \ldots, x_n$. If $m$ is a *monomial* (in the $x_i$'s), that is, if $m \in \mathcal{M}$, then

$$\rho_n(m) = \begin{cases} m & \text{if } m \in \mathcal{M}^n, \\ 0 & \text{otherwise .} \end{cases}$$

It follows that $\rho_n(\mathcal{M}) = \mathcal{M}^n \cup \{0\}$. Furthermore, every element $f \in R$ may be written $f = \sum_{m \in \mathcal{M}} c_m m$, with $c_m \in K$, and

$$\rho_n(f) = \sum_{m \in \mathcal{M}} c_m \rho_n(m) = \sum_{m \in \mathcal{M}^n} c_m m.$$

In what follows, we shall, when regarding an element $f \in R$ as a map $\mathcal{M} \to K$, write the value of the map on a particular monomial $m \in \mathcal{M}$ as $\mathrm{Coeff}(m, f)$. With this notation, any element $f \in R$ may be written $f = \sum_{m \in \mathcal{M}} \mathrm{Coeff}(m, f) m$.

Denote by $\mathcal{M}[n]$ the submonoid of $\mathcal{M}$ that is generated by the variables $\{x_{n+1}, x_{n+2}, x_{n+3}, \ldots\}$. Then, any $p \in \mathcal{M}$ may be written $p = p'p''$ with $p' \in \mathcal{M}^n$, $p'' \in \mathcal{M}[n]$. We will also need the notations $\mathcal{M}_d$ and $\mathcal{M}_d^n$ for the subset of monomials of total degree $d$ in $\mathcal{M}$ and $\mathcal{M}^n$, respectively. Viewing a monomial $m \in \mathcal{M}$ as a finitely supported function $\mathbb{N}^+ \to \mathbb{N}$, we denote by $\mathrm{Supp}(m) \subset \mathbb{N}^+$ its support, and by $\mathrm{maxsupp}(m)$ the maximal element in the support of $m$.

By means of the truncation maps, we define a *surjective system* (in the sense of [3, Chapter 1])

$$K \twoheadleftarrow K[[x_1]] \twoheadleftarrow K[[x_1, x_2]] \twoheadleftarrow K[[x_1, x_2, x_3]] \twoheadleftarrow \cdots \tag{6.3}$$

The surjective maps involved are of course the appropriate restrictions of the appropriate truncation maps. It is clear that the inverse limit of (6.3) is $R$.

Note that $\rho_n(R') = K[x_1, \ldots, x_n]$, and hence that

$$\rho_n(K[x_1, \ldots, x_{n+1}]) = K[x_1, \ldots, x_n].$$

Therefore, inside (6.3) one finds the following surjective system:

$$K \twoheadleftarrow K[x_1] \twoheadleftarrow K[x_1, x_2] \twoheadleftarrow K[x_1, x_2, x_3] \twoheadleftarrow \cdots \tag{6.4}$$

In (6.4) the maps are given by

$$K[x_1, \ldots, x_n] \simeq \frac{K[x_1, \ldots, x_n]}{(x_{n+1})} \twoheadleftarrow K[x_1, \ldots, x_{n+1}].$$

Since the functor $\varprojlim$ is left exact, the inverse limit of (6.4) can be isomorphically embedded in the inverse limit of (6.3), namely $R$. We call this ring ($K$-algebra, in fact) $\tilde{R}$. As we shall see in Lemma 6.3.5,

$$\tilde{R} \simeq \{ f \in R | \forall n : \rho_n(f) \in K[x_1, \ldots, x_n] \}.$$

If we consider the subset of the inverse limit of (6.4) consisting of those coherent sequences that have bounded degree, we get another K-algebra, which may be isomorphically embedded in $\tilde{R}$ and therefore in R. This K-algebra is nothing but the ring $R'$.

We will need some results from [75] on initial ideals and Gröbner bases in $R'$. First, we call any total order $>$ on $\mathcal{M}$ such that $(\mathcal{M}, >)$ is an ordered monoid (in the sense of [36]) with 1 as the smallest element, and such that $i < j \implies x_i > x_j$ an *admissible order on* $\mathcal{M}$. It is shown in [75] that if $>$ is an admissible order on $\mathcal{M}$, then for any $d$, any non-empty subset of $\mathcal{M}_d$ has a maximal element with respect to $>$. Therefore, any element $f \in R' \setminus \{0\}$ has a maximal or *leading* monomial which we denote by $\mathrm{Lpp}(f)$. For any ideal $I \subset R'$, the *initial ideal* (with respect to the chosen admissible order) is defined by

$$\mathrm{gr}(I) = \{\mathrm{Lpp}(f) | f \in I\}.$$

A subset $F \subset I$ such that the leading monomials of elements in F generate $\mathrm{gr}(I)$ is called a Gröbner basis for I. The technique for constructing such bases in $K[x_1, \ldots, x_n]$ is well understood [18, 21, 22, 11, 72, 71, 59].

In [75], a Gröbner basis theory for the ring $R'$ is developed. For technical reasons, we restrict our study to so-called *locally finitely generated* ideals:

**Definition 6.2.1.** A homogeneous ideal $I \subset R'$ is said to be locally finitely generated if it can be generated by a homogeneous set F such that, for all total degrees $d$, the subset $\{f \in F | |f| = d\}$ is finite (such a set is called *degree-finite*). Equivalently, we demand that

$$\forall d : \dim_K \frac{J_d}{\sum_{i=1}^{d} R'_i J_{d-i}} < \infty.$$

It is not hard to prove that the two conditions of the definition are equivalent (they are fulfilled, in particular, for homogeneous, finitely generated ideals). It is somewhat harder to see that for a locally finitely generated ideal I,

- The initial ideal $\mathrm{gr}(I)$ is locally finitely generated.
- I has a homogeneous, degree-finite Gröbner basis F.
- Any element $h \in I$ may be written as a (finite) combination $h = \sum_i f_i g_i$ where $f_i \in F$; this combination can be chosen to be *admissible* in the sense that
$$\forall i : \mathrm{Lpp}(f_i)\,\mathrm{Lpp}(g_i) \leq \mathrm{Lpp}(h).$$
- The set $F_{\leq d}$ is a *partial Gröbner basis up to degree* $d$ of I in the sense that if $h \in I_d$ then $h$ is a finite admissible combination of (the finitely many) elements in $F_{\leq d}$.

The locally finitely generated ideals are therefore an important and natural class of ideals. It is of interest, to study how well such an ideal $I$ is "approximated" by its truncations $\rho_n(I)$. In [76], we showed that the initial ideal $\mathrm{gr}(I)$ is determined by (the totality of all) the truncations $\rho_n(I)$. The main result of this article is, that $I$ itself is determined by its truncated ideals.

## 6.3 A topology on $R'$

We now set out to topologize $R'$. For a treatment of the topological concepts that we use, we refer to [58, 17, 23]. For filtrations and completions, we use the notations of [16].

The ideals

$$A_n := B_n \cap R' = \ker \rho_n : R' \to K[x_1, \ldots, x_n]$$

form a decreasing, separated and exhaustive filtration on $R'$ (we have that $A_0$ is the set of elements in $R'$ with non-zero constant term, and we define $A_{-1} = R'$). In what follows, we will not bother with $A_{-1}$, and let our indicies start at 0). With respect to the topology induced by this filtration, $R'$ is a Hausdorff topological ring. Recall (see for instance [16]) that the closure of any subset $M \subset R'$ is given by the formula

$$\bar{M} = \bigcap_{i=0}^{\infty} (M + A_n) \tag{6.5}$$

**Lemma 6.3.1.** *If* $I$ *is an ideal of* $R'$*, and* $h \in R'$*, then*

$$\bar{I} = \left\{ h \in R' \middle| \forall n : \rho_n(h) \in \rho_n(I) \right\} \tag{6.6}$$

*Proof.* Fix a positive integer $n$. If $h \in \bar{I}$ then $h \in I + A_n$, hence $\rho_n(h) \in \rho_n(I) + \rho_n(A_n) = \rho_n(I)$. Conversely, if $\rho_n(h) \in \rho_n(I)$ then there exists an $h' \in I$ such that $\rho_n(h) = \rho_n(h')$, whence $h - h' \in A_n$ and $h = h' + (h - h') \in I + A_n$. □

**Corollary 6.3.2.** *If* $I$ *is a closed ideal of* $R'$*, then for* $h \in R'$*,*

$$h \in I \iff \forall n : \ \rho_n(h) \in \rho_n(I).$$

**Corollary 6.3.3.** *If* $I, J$ *are closed ideals in* $R'$*, then*

$$I = J \iff \forall n : \ \rho_n(I) = \rho_n(J).$$

**Example 6.3.4.** The equivalence does not hold for general ideals $I, J \subset R'$. If $I = (x_1, x_2, x_3, \ldots)$ and $J = I + (x_1 + x_2 + x_3 + x_4 + \cdots)$ then $I \neq J$ but $\forall n : \rho_n(I) = \rho_n(J)$. In this example, neither $I$ nor $J$ are closed, both having $A_0$ as their closure.

**Lemma 6.3.5.** *The completion of* $R'$ *with respect to the* $A_n$*-filtration is isomorphic to the inverse limit of the following inverse system, where the surjective maps are the truncation homomorphisms* $\rho_n$*:*

$$K \twoheadleftarrow K[x_1] \twoheadleftarrow K[x_1, x_2] \twoheadleftarrow K[x_1, x_2, x_3] \twoheadleftarrow \cdots \tag{6.7}$$

*This inverse limit is isomorphic to the following subring of* $R$*:*

$$\tilde{R} = \{ f \in R | \forall n \in \mathbb{N} : \rho_n(f) \in K[x_1, \dots, x_n] \} \tag{6.8}$$

*Proof.* For the first part, it suffices to note that (6.7) is isomorphic to

$$R'/A_0 \twoheadleftarrow R'/A_1 \twoheadleftarrow R'/A_2 \twoheadleftarrow R'/A_3 \twoheadleftarrow \cdots \tag{6.9}$$

To prove the second part, one simply notes that an element $f$ of (6.8) defines a coherent sequence

$$(\rho_0(f), \rho_1(f), \rho_2(f), \rho_3(f), \dots),$$

and that any coherent sequence defines an element in (6.8) by

$$\sum_{n=0}^{\infty} \sum_{m \in \mathcal{M}^n} c_m m,$$

where $c_m$ is defined as the coefficient of $m$ in any sufficiently high component ($> n$) of the coherent sequence. □

Since $\tilde{R}$ is given the inverse limit topology, where the $K[x_1, \dots, x_n]$ are discretely topologized, and therefore Hausdorff, $\tilde{R}$ is a closed subspace of the infinite product space $\prod_{n=1}^{\infty} K[x_1, \dots, x_n]$. Furthermore, we have that $f_\nu \to f$ in $\tilde{R}$ iff $\rho_n(f_\nu) \to \rho_n(f)$ for all $n$. $R'$is given the subspace topology, and is a dense subset in $\tilde{R}$. A sequence of elements in $R'$ converges in $R'$ if, in addition, there is a global bound on the total degrees of the elements in the sequence.

We will briefly discuss one particular concept, namely what is meant by a convergent (infinite) sum $\sum_{l \in L} e_l$ in $R'$ and $\tilde{R}$, which we regard as Hausdorff topological groups. The set of finite subsets of $L$ form a directed set $\Delta$, and for any element $\delta \in \Delta$ we define $\phi(\delta) = \sum_{l \in \delta} e_l$. This is a finite sum, hence it is well-defined. We then say that the sum is convergent (the family $(e_l)_{l \in L}$ is *summable*) if the net $(\Delta, \phi)$ converges to an element $f$ in the group. This means that for any neighborhood $U$ of $f$, the net is residual in $U$, which means that there exists a $\delta_0 \in \Delta$ such that for $\delta > \delta_0$ we have that $\phi(\delta) \in U$. For $\tilde{R}$, which is a complete topological group, we can apply the so-called Cauchy criterion [17, III, chapter 6]: the sequence is summable iff for each neighborhood $V$ of zero, there is a finite subset $L' \subset L$ such that $\sum_{l \in L''} e_l \in V$ for all finite subsets $L''$ which do not meet $L'$. Combining this with a proposition on sequences in Hausdorff, complete groups topologized by filtrations [16, III, chapter 2.6, prop 5] we get:

**Lemma 6.3.6.** *The sum* $\sum_{l \in L} e_l$, *with* $e_l \in \tilde{R}$, *converges iff* $e_l \to 0$ *with respect to the filter of finite complements of subsets of* $L$.

*If in addition the* $f_\nu$*'s have a common bound on their degrees, then the limit lies in* $R'$.

We remark that the fact that $0 \in \tilde{R}$ has a countable system of neighborhoods (namely, the $A_n$'s) implies [17, III, chapter 6] that any convergent sum is at most countable. We also have that [17, III, chapter 5.4] $\sum_{l \in L} e_l$ converges iff $\sum_{l \in L} \rho_n(e_l)$ converges in $K[x_1, \ldots, x_n]$ for all $n$. Since $K[x_1, \ldots, x_n]$ has the discrete topology, we get that $\sum_{l \in L} e_l$ converges iff, for all $n$, all but finitely many of the $\rho_n(f_l)$ are zero. This fact is used in the proof of the next lemma.

**Proposition 6.3.7.** *The sum* $\sum_{l \in L} f_l$, $f_l \in R'$, *converges to an element* $f \in R'$ *if the following two conditions hold:*

1. $\exists d : \forall l : |f_l| \leq d$,
2. $\forall m \in \mathcal{M}$ : *the set* $\{ l \in L | m \in \mathrm{Mon}(f_l) \}$ *is finite.*

*Proof.* If $\sum_{l \in L} f_l$ converges to an element in $\tilde{R}$, this element must have total degree $\leq d$, and hence, it must lie in $R'$. It is therefore enough to show that for all $n$, $\rho_n(f_l) = 0$ for almost all $l$. For a fixed $n$ we have that $\mathrm{Mon}(\rho_n(f_l)) \subset \mathcal{M}^n_{\leq d}$, which is a finite set. Therefore, since each of the finitely many $m \in \mathcal{M}^n_{\leq d}$ may occur only finitely many times as an element of $\mathrm{Mon}(\rho_n(f_l))$, we must have that all but finitely many of the sets $\mathrm{Mon}(\rho_n(f_l))$ are empty. This is the desired conclusion. □

*Remark 6.3.8.* The converse does not hold: the sum $\sum_{n=1}^{\infty} \left( x_n^n - x_{n-1}^{(n-1)} \right)$ converges to zero in $R'$, yet there is no common bound of the total degrees of the terms of the sum.

**Proposition 6.3.9.** *For any ideal* $I \subset R'$, *the closure* $\bar{I} \subset R'$ *is equal to the set of convergent sums (in* $R'$*) of elements in* $I$.

*Proof.* If $f_l \in I$ for all $l \in L$, and if $R' \ni f = \sum_{l \in L} f_l$, then for each finite subset $L' \subset L$ the corresponding partial sum $\sum_{l \in L'} f_l$ belongs to I. It is observed in [17, III, chapter 5.3] that a convergent sum is contained in the closure of the set of all finite partial sums. This closure is a subset of the closure of I, hence the assertion.

Conversely, if $f \in \bar{I}$ then for all $n$, $\rho_n(f) \in \rho_n(I)$, which implies that

$$f = \sum_{n=0}^{\infty} (\rho_n(f) - \rho_{n-1}(f)) \,, \quad \rho_{-1}(f) = 0$$

is a convergent sum of elements in I. The convergence is guaranteed by the fact that

$$\mathrm{Mon}(\rho_n(f) - \rho_{n-1}(f)) \subset \left(\mathcal{M}^n \setminus \mathcal{M}^{n-1}\right),$$

which means that the terms have disjoint sets of occuring monomials, and hence that Proposition 6.3.7 applies. □

**Lemma 6.3.10.** *The ideals* $A_n$ *are prime ideals.*

*Proof.* $\frac{R'}{A_n} = K[x_1, \ldots, x_n]$ which is a domain. □

**Corollary 6.3.11.** *The monoid ideals* $A_n \cap \mathcal{M}$ *have the following property:* $p, q \in \mathcal{M} \setminus A_n \implies pq \in \mathcal{M} \setminus A_n$.

**Lemma 6.3.12.** *The ideals* $A_n$ *are "pseudo monomial ideals" in the sense that for any* $f \in R'$, $f \in A_n \iff \mathrm{Mon}(f) \subset A_n$.

*Proof.* $\rho_n(f) = 0 \iff \forall m \in \mathrm{Mon}(f) : \rho_n(m) = 0$. □

*Remark 6.3.13.* $A_n$ is not generated by monomials, so it is not a "monomial ideal". It is however the closure of the monomial ideal

$$(x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \ldots).$$

Recall the definition of admissible order on $\mathcal{M}$. We consider now the following admissible order $>_{\mathrm{rlex}}$, the so-called *reverse lexicographic order* (or graded reverse lexicographic order). Let $m, m' \in \mathcal{M}$, then we can write $m = \prod_{i=1}^{\infty} x_i^{\alpha_i}$ and $m' = \prod_{i=1}^{\infty} x_i^{\beta_i}$ where $\alpha_i = \beta_i = 0$ for almost all $i$. Then $m >_{\mathrm{rlex}} m'$ iff $|m| > |m'|$, or if $|m| = |m'|$ and $\alpha_j < \beta_j$ for the largest $j$ such that $\alpha_j \neq \beta_j$.

It is not hard to see that any admissible order $>$ on $\mathcal{M}$ restricts to an admissible order (in the usual sense, c.f [70], where the synonym *term order* is used, or [25], where such a total order is called a *monomial order*) on $\mathcal{M}^n$ for all $n$. The restriction to $\mathcal{M}^n$ of the reverse lexicographic order gives the "usual" reverse lexicographic order (as defined in e.g [25]).

**Lemma 6.3.14.** *For any* $d$*, the homogeneous component of degree* $d$ *of the monoid ideal* $A_n \cap \mathcal{M} = \langle \mathcal{M}[n] \rangle$ *is a (terminal) reverse lexicographic segment*[1] *in* $\mathcal{M}_d$ *in the following sense: if* $m \in A_n \cap \mathcal{M}_d$ *and* $p \in \mathcal{M}_d$*, then if* $p \leq_{\mathrm{rlex}} m$ *then* $p \in A_n \cap \mathcal{M}_d$*. In fact,*

$$A_n \cap \mathcal{M}_d = \left\{ p \in \mathcal{M}_d \middle| p <_{\mathrm{rlex}} x_n^d \right\} \tag{6.10}$$

[1] This terminology is inspired by [73], where the term *lexicographical segment* is used for a subset of $\mathcal{M}_d^n$ consisting of all elements that are lexicographically smaller than some monomial in $M_d^n$.

*Proof.* It suffices to show (6.10). Let $m \in A_n \cap \mathcal{M}$, then there exists a $j > n$ such that $x_j \,|m$. If $|m| = d$ then $m <_{rlex} x_n^d$, by the definition of the reverse lexicographic order. Conversely, a monomial $m$ of degree $d$ is reverse lexicographically smaller than $x_n^d$ iff it is of the form $m = \prod_{i=1}^{\infty} x_i^{\alpha_i}$ where $\sum_{i=1}^{\infty} \alpha_i = d$ and $\alpha_j > 0$ for some $j > n$. Therefore, $\rho_n(m) = 0$, and $m \in A_n$. $\square$

*Remark 6.3.15.* The lemma implies that the finite set

$$\mathcal{M}_d^n = \mathcal{M}_d \setminus (\mathcal{M}_d \cap A_n)$$

is an initial reverse lexicographic segment in $\mathcal{M}_d$.

**Lemma 6.3.16.** *Let $>$ be the reverse lexicographic order, let $n$ be any positive integer, and let $f \in R'$ be homogeneous of degree $d$. Then $f \in A_n$ iff* $\mathrm{Lpp}(f) \in A_n$.

*Proof.* By Lemma 6.3.12, it suffices to show that $\mathrm{Lpp}(f) \in A_n$ iff $\mathrm{Mon}(f) \subset A_n$. Since $\mathrm{Lpp}(f) \in \mathrm{Mon}(f)$, one direction is clear. Suppose therefore that $\mathrm{Lpp}(f) \in A_n \cap \mathcal{M}_d$ and let $m \in \mathrm{Mon}(f) \subset \mathcal{M}_d$. Since $m \leq_{rlex} \mathrm{Lpp}(f)$, and since $A_n \cap \mathcal{M}_d$ is a terminal reverse lexicographic segment, we have that $m \in A_n$. $\square$

*Remark 6.3.17.* The corresponding result for the "ordinary" reverse lexicographic order is well-known, and mentioned for instance in [25, Proposition 15.4 c].

## 6.4 Ideals that are locally finitely generated are closed

**Lemma 6.4.1.** *If $J$ is a locally finitely generated ideal in $R'$, then for any $d$ there exists an $N$ such that for $n > N$,*

$$(J \cap A_n)_d = (JA_n)_d.$$

*Proof.* Since $PQ \subset P \cap Q$ for any ideals $P, Q$, we need only show that $(J \cap A_n)_d \subset (JA_n)_d$ for sufficiently large $n$. Choose a partial Gröbner basis of $J$ up to degree $d$, with respect to the reverse lexicographic term order. Denote by $\{f_{ij}\}_{1 \leq j \leq r_i}$ the Gröbner basis elements of degree $i$, for $1 \leq i \leq d$. Let

$$N = \max_{\substack{1 \leq i \leq d \\ 1 \leq j \leq r_i}} (\mathrm{maxsupp}\,(\mathrm{Lpp}(f_{ij})))$$

and let $n > N$. Then $\mathrm{Lpp}(f_{ij}) \notin A_n$, and hence $f_{ij} \notin A_n$, by Lemma 6.3.16.

Now take $h \in J_d \cap A_n$. Then $h$ may be written as an admissible combination

$$h = \sum_{i=1}^{d} \sum_{j=1}^{r_i} g_{ij} f_{ij}\ , \qquad \mathrm{Lpp}(f_{ij})\,\mathrm{Lpp}(g_{ij}) \leq_{rlex} \mathrm{Lpp}(h) \qquad (6.11)$$

We may without loss of generality assume that $g_{ij}$ is homogeneous of degree $d - |f_{ij}| = d - i$ for all $i, j$.

Since (6.11) is an admissible combination, we have that

$$\mathrm{Lpp}(f_{ij})\,\mathrm{Lpp}(g_{ij}) \leq_{\mathrm{rlex}} \mathrm{Lpp}(h) \in A_n,$$

hence that $\mathrm{Lpp}(f_{ij})\,\mathrm{Lpp}(g_{ij}) \in A_n$, hence that $\mathrm{Lpp}(g_{ij}) \in A_n$, hence that $g_{ij} \in A_n$. We have used Lemma 6.3.14, Corollary 6.3.11 and Lemma 6.3.16.

In this argument, $i, j$ were arbitrary, so $g_{ij} \in A_n$ for all $i, j$. This implies (by (6.11), since $f_{ij} \in J$) that $h \in JA_n$. □

**Theorem 6.4.2.** *Locally finitely generated ideals in $R'$ are closed ideals.*

*Proof.* Let $J \subset R'$ be locally finitely generated, and let $h \in \bar{J}$. We must show that $h \in J$. Without loss of generality, we can assume that $h$ is homogeneous of degree $d$. Let $\{f_1, \ldots, f_r\}$ be a homogeneous generating set for the ideal generated by $J_{\leq d}$. Using Lemma 6.4.1, we get that there exists an $N$ such that for $n > N$, $(J \cap A_n)_d = (JA_n)_d$. Since $h \in \bar{J} = \cap_{i=1}^{\infty} (J + A_i)$ , we can write

$$h = t_n + s_n = t_{n+1} + s_{n+1} = t_{n+2} + s_{n+2} = \cdots \tag{6.12}$$

with $t_i \in J_d$, $s_i \in (A_i)_d$. We have that

$$s_{k+1} - s_k \in (J \cap A_k)_d = (JA_k)_d$$

whenever $k \geq n$. Therefore, we can write

$$s_{k+1} - s_k = \sum_{i=1}^{r} f_i g_{ik}$$

whenever $k \geq n$, with $g_{ik} \in A_k$. By Lemma 6.3.6 we may form the telescoping sum

$$-s_n = \sum_{k=n}^{\infty} (s_{k+1} - s_k) = \sum_{k=n}^{\infty} \sum_{i=1}^{r} f_i g_{ik} = \sum_{i=1}^{r} f_i \sum_{k=n}^{\infty} g_{ik} \tag{6.13}$$

It follows from Lemma 6.3.6 that $\sum_{k=n}^{\infty} g_{ik}$ converges to an element in $R'$. The sums

$$\sum_{(k,i) \in [n,\infty) \times [i,r]} f_i g_{ik} = \sum_{k=n}^{\infty} \sum_{i=1}^{r} f_i g_{ik}$$

are convergent by Proposition 6.3.7. Hence, the rearrangement of sums in (6.13) is justified (see [17, III, chapter 6]). Therefore, $s_n \in J$. From (6.12) we conclude that $h \in J$. □

Using Corollary 6.3.2 and Corollary 6.3.3, we get

**Corollary 6.4.3.** *If* $J \subset R'$ *is locally finitely generated, then* $h \in R'$ *belongs to* $J$ *iff* $\rho_n(h) \in \rho_n(J)$ *for all positive integers* $n$*.*

This means that a locally finitely generated ideals is determined by its truncated ideals.

## 6.5 Closedness of ideals generated by monomials

**Proposition 6.5.1.** *If* $I \subset R'$ *is an ideal generated by monomials, then for* $f \in R'$*, we have that* $\mathrm{Mon}(f) \subset I \iff f \in \bar{I}$*.*

*Proof.* If $\mathrm{Mon}(f) \subset I$ then $f = \sum_{m \in \mathrm{Mon}(f)} \mathrm{Coeff}(m, f)m$ is a convergent sum of elements in I. Hence, $f \in \bar{I}$.

Conversely, if $f \in \bar{I}$ then for all $n$, $\rho_n(f) \in \rho_n(I)$. The latter ideal is a monomial ideal in $K[x_1, \dots, x_n]$. From the well-known property of such ideals, we get that $\mathrm{Mon}(\rho_n(f)) \subset \rho_n(I)$. Since I is a monomial ideal, $\rho_n(I)$ may be regarded as a subideal of I. We conclude that $\mathrm{Mon}(\rho_n(f)) \subset I$. Noting that $\mathrm{Mon}(f) = \cup_{n=0}^{\infty} \mathrm{Mon}(\rho_n(f))$, we get that $\mathrm{Mon}(f) \subset I$. □

**Lemma 6.5.2.** *If* $J$ *is generated by monomials, then* $J$ *is closed iff it is locally finitely generated.*

*Proof.* We already know that locally finitely generated ideals are closed. Conversely, suppose that J is closed. Define

$$\mathrm{mingen}(J) = \{ m \in J \cap \mathcal{M} \mid \nexists s, t \neq 1, s \in J : m = st \},$$

then J is locally finitely generated iff $\mathrm{mingen}(J)_d$ is finite for all $d$. By induction, we assume that $\mathrm{mingen}(J)_{<d}$ is finite. Choose an index set S such that $\mathrm{mingen}(J) = \{ m_i | i \in S \}$, and put $f = \sum_{i \in S} m_i$. Then $f \in \bar{J} = J$, by Proposition 6.5.1 and the fact that J is closed. Since $|f| = d$ we must have that

$$f = \sum_{k=1}^{r} p_k g_k + \sum_{i \in S'} c_i m_i \tag{6.14}$$

where $p_k \in \mathrm{mingen}(J)_{<d}$, $g_k \in R'$, $c_i \in K$ and $S' \subset S$ is finite. Furthermore, we can assume that for all $k$, $g_k$ is homogeneous of degree $d - |p_k|$.

Now pick a monomial $m \in \mathcal{M}_d$, and study the corresponding multihomogeneous component in (6.14). If $m \in \mathrm{mingen}(J)$, then $m$ occurs in the left hand side, with coefficient 1, hence must occur in the right hand side. It can not be that $m \in \mathrm{Mon}(\sum_{k=1}^{r} p_k g_k)$, since then we would have that $m \in p_k \mathrm{Mon}(g_k)$ for some $k$, hence that $m = p_k v$ for some $v \in \mathrm{Mon}(g_k)$. This contradicts the fact that $m \in \mathrm{mingen}(J)$. Therefore, $m$ must occur in $\sum_{i \in S'} c_i m_i$, so that if $m = m_i$ then $i \in S'$ and $c_i = 1$.

If on the other hand $m \notin \mathrm{mingen}(J)$ then $m$ does not occur in the left hand side, so it must cancel in the right hand side. It does not occur in $\sum_{i \in S'} c_i m_i$, so it must cancel in $\sum_{k=1}^{r} p_k g_k$.

Putting this together, we see that we must have that $\sum_{k=1}^{r} p_k g_k = 0$, that $S' = S$, and that $c_i = 1$ for all $i \in S'$. Therefore, $S$ is finite, so $J$ is locally finitely generated. □

## 6.6 Ideals with locally finitely generated associated homogeneous ideal are closed

In [80] the Gröbner basis theory for locally finitely generated ideals, developed in [75], is extended to non-homogeneous ideals with locally finitely generated associated homogeneous ideal. We mean by the associated homogeneous ideal $\mathrm{gr}_{\mathcal{T}}(I)$ of an ideal $I \subset R'$ the associated graded ideal with respect to the (increasing) total-degree filtration on $R'$, or, in other words,

$$\mathop{\mathrm{gr}}_{\mathcal{T}}(I) = \{\, c(f) \mid f \in I \,\},$$

where $c(f)$ denotes the homogeneous component of maximal degree of $f$. We show in [80] that if $\mathrm{gr}_{\mathcal{T}}(I)$ is locally finitely generated, then $I$ has a Gröbner basis that is locally filtered finite, that is, contains but finitely many (in general non-homogeneous) elements of any total degree.

Furthermore, we observe that the reverse lexicographic termorder is *degree-compatible*, which means that it refines the partial order given by

$$m \geq m' \iff |m| \geq |m'|.$$

It is not hard to see[2] that for such a termorder, $\mathrm{Lpp}(f) = \mathrm{Lpp}(c(f))$ for any $f \in R'$, and furthermore that $\mathrm{gr}(I) = \mathrm{gr}(c(I))$.

**Lemma 6.6.1.** *If* $\mathrm{gr}_{\mathcal{T}}(I)$ *is locally finitely generated, then*

$$\forall d : \exists N(d) : \forall n > N(d) : \quad (I \cap A_n)_{\leq d} = (IA_n)_{\leq d} \tag{6.15}$$

[2] It is proved for $R'$ in [80], and it is a trivial generalization of the well-known result for polynomial rings. A variation of this fact is mentioned in [81, Proposition 1.8].

*Proof.* The inclusion $IA_n \subset I \cap A_n$ always hold, hence so does $(IA_n)_{\leq d} \subset (I \cap A_n)_{\leq d}$ . We concentrate on the reverse inclusion.

Using the results of [80] we assume the existence of a locally filtered finite Gröbner basis $F$ of $I$, with respect to the reverse lexicographic order. Fix a $d$, and let $F_{\leq d}$ be the (finite) set of all elements in $F$ of total degree no greater than $d$. Choose $n$ so large that no $\mathrm{Lpp}(f_i) \in A_n$. Let $h \in (I \cap A_n)_{\leq d}$. In particular, $\mathrm{Lpp}(h) \in A_n$.

Since every element in $I_{\leq d}$ may be written as an admissible combination of elements in $F_{\leq d}$, we can write

$$h = \sum_i f_i g_i, \quad f_i \in F_{\leq d},\, g_i \in R', \quad \mathrm{Lpp}(h) \geq \mathrm{Lpp}(f_i g_i) \tag{6.16}$$

where, since $>$ is degree-compatible, we have that $|h| \geq |f_i g_i|$.

We now prove that for $i$ such that $|h| = |f_i g_i|$, we have that $c(g_i) \in A_n$. Since $\mathrm{Lpp}(h) \in A_n$, and since $\mathrm{Lpp}(f_i g_i) \leq \mathrm{Lpp}(h)$, then if $|h| = |f_i g_i|$ then $\mathrm{Lpp}(f_i g_i) \in A_n$ by Lemma 6.3.14. Since $A_n$ is a prime ideal, and since $\mathrm{Lpp}(f_i) \notin A_n$, we have that $\mathrm{Lpp}(g_i) \in A_n$. By Lemma 6.3.16 this gives that $c(g_i) \in A_n$.

It follows that we can write

$$\begin{aligned} h &= \sum_{|f_i g_i| = |h|} f_i g_i + \sum_{|f_i g_i| < |h|} f_i g_i && (6.17) \\ &= \sum_{|f_i g_i| = |h|} f_i c(g_i) + \sum_{|f_i g_i| = |h|} f_i (g_i - c(g_i)) + \sum_{|f_i g_i| < |h|} f_i g_i && (6.18) \end{aligned}$$

where the first sum of (6.18), as we have shown, is in $(IA_n)_{\leq d}$, and where the remaining sums have total degree $< d$. It is immediate from (6.18) that

$$U := \left( \sum_{|f_i g_i| = |h|} f_i (g_i - c(g_i)) + \sum_{|f_i g_i| < |h|} f_i g_i \right) \in (I \cap A_n)_{\leq d-1} .$$

By induction (we assume that we have chosen $n$ large enough in the previous step), we can assume that $U \in (IA_n)_{\leq d-1}$. □

With this lemma, the proof of the next theorem is almost identical to the proof of Theorem 6.4.2, and is omitted.

**Theorem 6.6.2.** *If* $\mathrm{gr}_{\mathcal{T}}(I)$ *is locally finitely generated, then* $I$ *is closed.*

**Question 6.6.3.** *Are finitely generated ideals closed?*

Note that if finitely generated ideals always have locally finitely generated associated homogeneous ideals, then the answer to the above question is "yes". The author has not been able to prove this very plausible conjecture.

## 6.7 Distributive lattices of ideals

We refer to [37, 14] for definitions and standard results on lattices.

Let $f_1, f_2, \dots, f_r \in R'$ be homogeneous. Then, the principal ideals $(f_1)$ to $(f_r)$ are elements in the lattice of ideals in $R'$, where the lattice operations $\vee$ and $\wedge$ are $+$ and $\cap$. Denote this lattice by $L$, and denote by $F$ the sub-lattice generated by the principal ideals $(f_1)$, $(f_2)$, $\dots$, $(f_r)$. Similarly, denote, for any positive integer $n$, by $L_n$ the lattice of ideals in $K[x_1, \dots, x_n]$. Note that $F$, $L$ and $L_n$ are modular.

**Lemma 6.7.1.** *For any positive integer $n$, the $K$-algebra homomorphism*

$$\rho_n : R' \to K[x_1, \dots, x_n]$$

*induces (by extension of ideals) a surjective map ${\rho_n}^e : L \to L_n$ by ${\rho_n}^e(I) = \rho_n(I)$. This map has the properties that ${\rho_n}^e(I + J) = {\rho_n}^e(I) + {\rho_n}^e(J)$ (it is a* join-homomorphism*), and ${\rho_n}^e(I \cap J) \subset {\rho_n}^e(I) \cap {\rho_n}^e(J)$.*

*Proof.* Since $\rho_n$ is surjective, the extension of $I$ is simply the image $\rho_n(I)$; furthermore, every ideal in $K[x_1, \dots, x_n]$ is an extended ideal. For extensions, the relations $(a+b)^e = a^e + b^e$ and $(a \cap b)^e \subset a^e \cap b^e$ always hold (see for instance chapter 1 of [3]). $\square$

*Remark 6.7.2.* ${\rho_n}^e$ is no lattice homomorphism. Consider $a = (x_1 + x_2)$ and $b = (x_1 + 2x_2)$. Then $a \cap b = (x_1^2 + 3x_1x_2 + 2x_2^2)$, so that $\rho_1(a \cap b) = (x_1^2)$. On the other hand, $\rho_1(a) \cap \rho_1(b) = (x_1) \cap (x_1) = (x_1)$.

We shall use the following two facts from [78]: first, that $R'$ is a unique factorization domain, so that lcm's and gcd's of finite tuples are defined; secondly, that lcm's commute with the truncation homomorphisms in the following way:

**Lemma 6.7.3.** *Suppose that $g_1, \dots, g_s \in R' \setminus K$ are homogeneous. Then, for all sufficiently large integers $n$ we have that*

$$\operatorname{lcm}(\rho_n(g_1), \dots, \rho_n(g_s)) = \rho_n(\operatorname{lcm}(g_1, \dots, g_s))$$
$$|\operatorname{lcm}(\rho_n(g_1), \dots, \rho_n(g_s))| = |\operatorname{lcm}(g_1, \dots, g_s)|.$$

*The corresponding results for gcd's also holds.*

The following simple lemma and its corollary will be of great use to us:

**Lemma 6.7.4.** *Suppose that $U$ is a distributive lattice generated by the elements $u_1, \dots, u_r$, and that $V$ is a modular lattice. Let $f : U \to V$ be a map with the following properties:*

*(A)* $\forall a, b \in U : f(a \vee b) = f(a) \vee f(b)$ *($f$ is a join-homomorphism),*

*(B) If $m_i$ and $m_j$ are (finite) meets of elements in $\{u_1, \dots, u_r\}$, then $f(m_i \wedge m_j) = f(m_i) \wedge f(m_j)$.*

*Denote by $\langle f(U) \rangle$ the smallest sublattice of $V$ containing $f(U)$ (in other words, the sublattice generated by $f(U)$). Then the following are equivalent:*

*(i) $f$ is a lattice homomorphism,*

*(ii) The lattice $\langle f(U) \rangle$ is distributive.*

*If the conditions hold true, then $f(U) = \langle f(U) \rangle$.*

*Proof.* $(i) \implies (ii)$: A homomorphic image of a distributive lattice is distributive. Therefore, $f(U)$ is a distributive lattice, and hence $\langle f(U) \rangle = f(U)$.

$(ii) \implies (i)$: We must verify that for all $a, b \in U$, $f(a \wedge b) = f(a) \wedge f(b)$. Since $U$ is distributive and generated by $\{u_1, \dots, u_r\}$, we can write $a = \bigvee_{j=1}^{t} m_j$ and $b = \bigvee_{k=1}^{s} p_k$, where the $m_j$'s and the $p_k$'s are finite meets of elements in $\{u_1, \dots, u_r\}$. Then

$$\begin{aligned}
f(a \wedge b) &= f(\vee_{j=1}^{t} m_j \bigwedge \vee_{k=1}^{s} p_k) \\
&= f(\bigvee_{j,k} m_j \wedge p_k) && \text{since } U \text{ is distributive} \\
&= \bigvee_{j,k} f(m_j \wedge p_k) && \text{by (A)} \\
&= \bigvee_{j,k} f(m_j) \wedge f(p_k) && \text{by (B)} \\
&= \vee_{j=1}^{t} f(m_j) \bigwedge \vee_{k=1}^{s} f(p_k) && \text{since } \langle f(U) \rangle \text{ is distributive} \\
&= f(\vee_{j=1}^{t} m_j) \bigwedge f(\vee_{k=1}^{s} p_k) && \text{by (A)} \\
&= f(a) \wedge f(b)
\end{aligned}$$

□

**Corollary 6.7.5.** *If $F$ is distributive, then $\langle \rho_n(F) \rangle$ is distributive for almost all $n$ iff ${\rho_n}^e$ is a lattice homomorphism for almost all $n$. Furthermore, for almost all $n$, $\langle \rho_n(F) \rangle$ is distributive iff ${\rho_n}^e$ is a lattice homomorphism.*

*Proof.* We have that

$$(f_{i_1}) \cap \cdots \cap (f_{i_s}) = (\mathrm{lcm}(f_{i_1}, \dots, f_{i_s})).$$

Furthermore, the least common multiple commutes with the truncation homomorphisms for almost all $n$, by Lemma 6.7.3. Combining this result with Lemma 6.7.1, we see that the requirements of Lemma 6.7.4 are fulfilled. □

**Proposition 6.7.6.** *A modular lattice generated by a family of* $r$ *elements* $x_1, \ldots, x_r$ *is distributive iff all of the so-called JMB conditions*[3] *hold for the family and all of its sub-families. The* $s$*'th JMB condition, for* $1 \leq s \leq r-2$ *is fulfilled if for each* $\sigma \in S_r$*, the symmetric group on* $r$ *letters, we have that*

$$\left(\bigwedge_{t=1}^{s} x_{\sigma t}\right) \wedge \left(\bigvee_{u=s+1}^{r} x_{\sigma u}\right) = \bigvee_{u=s+1}^{r} \left(x_{\sigma u} \wedge \bigwedge_{t=1}^{s} x_{\sigma t}\right) \tag{6.19}$$

*Proof.* It is clear that (6.19) is necessary. In [47], Jónsson proves that (6.19) is sufficient. □

**Lemma 6.7.7.** *Suppose that for each positive integer* $n$*, the sub-lattice* $\langle \rho_n(F) \rangle$ *of* $L_n$ *is distributive. Suppose furthermore that for each finite family* $\{S_1, \ldots, S_\nu\}$ *of subsets*

$$S_j = \{s_{j,1}, \ldots, s_{j,q_j}\} \subset \{1, \ldots, r\},$$

*the ideal* $\sum_{j=1}^{\nu} \left(\cap_{i=1}^{q_j} (f_{s_{j,i}})\right)$ *is closed. Then the lattice* $F \subset L$ *is distributive.*

*Proof.* We must show that $F$ fulfills the JMB condition (6.19). Without loss of generality, we can assume that $\sigma$ is the identity. Then, the left hand side of (6.19) translates to

$$\mathsf{LHS} = \left(\mathrm{lcm}(f_1, \ldots, f_s)\right) \cap \left(f_{s+1}, \ldots, f_r\right) \tag{6.20}$$

whereas the right hand side becomes the (finitely generated, homogeneous) closed ideal

$$\mathsf{RHS} = \left(\mathrm{lcm}(f_{s+1}, f_1, \ldots, f_s), \mathrm{lcm}(f_{s+2}, f_1, \ldots, f_s), \ldots, \mathrm{lcm}(f_r, f_1, \ldots, f_s)\right) \tag{6.21}$$

The inclusion RHS $\subset$ LHS holds for general reasons: each generator of the RHS is divisible by $w := \mathrm{lcm}(f_1, \ldots, f_s)$, since it is an least common multiple of $w$ and some $f_{s+b}$; each element of the RHS may be written as

$$\sum_{i=s+1}^{r} e_j \, \mathrm{lcm}(w, f_i) = \sum_{i=s+1}^{r} e_j \frac{\mathrm{lcm}(w, f_i)}{f_i} f_i \in (f_{s+1}, \ldots, f_r),$$

where $e_j \in R'$ may be taken to be homogeneous.

By our assumptions, for all $n$, we have that $\rho_n(F)$ is distributive. Hence $\rho_n(\mathsf{LHS}) = \rho_n(\mathsf{RHS})$, and in particular, $\rho_n(\mathsf{LHS}) \subset \rho_n(\mathsf{RHS})$. Since the RHS is closed, we conclude using Corollary 6.3.2 that LHS $\subset$ RHS. □

We have previously shown that locally finitely generated, and in particular, homogeneous and finitely generated ideals are closed (Theorem 6.4.2). Therefore, we have in fact proved:

[3] JMB stands for Jónsson [47], Musti and Buttafuoco [57]. The terminology is taken from [5].

**Theorem 6.7.8.** *If for all positive integers $n$, the sublattice $\langle \rho_n(F) \rangle$ of $L_n$ is distributive, then so is the lattice $F$.*

*Remark 6.7.9.* In this case, $F$ is finite. It is also immediate that each element of $F$ is a homogeneous, finitely generated ideal. By Corollary 6.7.5, we get that ${\rho_n}^e$ is a lattice homomorphism for all $n$.

The following questions remain:

**Question 6.7.10.** *If $F$ is distributive, is then $\langle \rho_n(F) \rangle$ distributive for almost all $n$?*

An affirmative answer would, among other things, yield a sharper version of a proposition in [78] about when the "Hilbert Numerators" are polynomials.

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Alas, what are you after all, my written and painted thoughts! It was not long ago that you were still so colorful, young, and malicious, full of thorns and secret spices – you made me sneeze and laugh – and now? You have already taken off your novelty, and some of you are ready, I fear, to become truths: they already look so immortal, so pathetically decent, so dull!

*Friedrich Nietzsche*