Number theory, Talteori 6hp, Kurskod TATA54, Provkod TEN1
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Each problem is worth 3 points. To receive full points, a solution needs to be complete. Indicate which theorems from the textbook that you have used, and include all auxillary calculations.

No aids, no calculators, tables, nor texbooks.

1) Find all odd positive integers n such that $n+1$ is divisible by 3 and $n+2$ is divisible by 5.

2) Show that the congruence

$$x^3 + x + 1 \equiv 0 \mod 11^n$$

has a unique solution for every positive integer n.

3) The number 431 is a prime. Determine if the congruence

$$2x^2 - 6x + 38 \equiv 0 \mod 431$$

has any solutions.

4) How many primitive roots are there mod 5? Find them all. How many primitive roots are there mod 25? For each primitive root a mod 5 that you find, check which of the "lifts"

$$a + 5t, \qquad 0 \leq t \leq 4$$

are primitive roots mod 25.

5) Determine the (periodic) continued fraction expansion of $\sqrt{7}$. Determine the solution $(x, y) \in \boldsymbol{Z}^2$, $x, y > 0$, to $x^2 - 7y^2 = 1$ with smallest x.

6) For each positive integer n, let $g(n)$ denote the number of triples (a, b, c) of positive integers such that $abc = n$. Calculate $g(p^e)$, with p a prime, then show that g is a multiplicative arithmetic function and use this to give a formula for $g(n)$ in terms of the prime factorisation of n.

(Hint: the number-of-divisors function τ is the Dirichlet square of the constant-one function. What is the Dirichlet cube?).