Number theory, Talteori 6hp, Kurskod TATA54, Provkod TEN1
June 4, 2019
LINKÖPINGS UNIVERSITET
Matematiska Institutionen
Examinator: Jan Snellman

All problems are worth 3 points. To receive full points, a solution needs to be complete. Prove your assertions, indicate which theorems from the textbook that you have used, and include all auxillary calculations.

No aids, no calculators, tables, nor textbooks.

1) Find all $(x, y) \in \mathbf{Z}^2$ such that (x, y) is a solution to $3x - 7y = 1$, and x, y are relatively prime.

2) Write, if possible, 6! as a sum of two squares.

3) Show that

$$\frac{10}{7} < \sqrt[3]{3} < \frac{13}{9} < \frac{3}{2}$$

and that if

$$\frac{10}{7} < \frac{a}{b} < \sqrt[3]{3} < \frac{c}{d} < \frac{3}{2}$$

with $a, b, c, d \in \mathbf{N}$ then $b > 7$, $d > 2$.

4) $(x, y) = (10, 3)$ is a positive solution to Pell's equation $x^2 - 11y^2 = 1$. Find another!

5) Let $f(x) = x^2 - x + 1$. Show that, modulo 7, both zeroes of $f(x)$ are primitive roots. Determine the number of zeroes of $f(x)$ modulo 7^n for all $n \geq 2$.

6) Define the arithmetical function f by

$$f(n) = \sum_{d|n} \frac{\mu(d)}{d},$$

where μ is the Möbius function. Is f multiplicative? Denote by $\mathrm{Supp}(n)$ the set of primes dividing n. Does the value of $f(n)$ depend only on $\mathrm{Supp}(n)$?

7) Show that the polynomial $f(x) = x^4 + 1$ does not factor over $\mathbf{Z}$, i.e., can not be written as a product $f(x) = a(x)b(x)$ with both $a(x), b(x)$ of lower degree, yet $f(x)$ factors modulo any prime!

(Hint: consider the cases $p = 2$, $p \equiv 1, 5 \mod 8$, $p \equiv 7 \mod 8$, $p \equiv 3 \mod 8$)