Number theory, Talteori 6hp, Kurskod TATA54, Provkod TEN1
Nov 1, 2019
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SOLUTIONS

1) If a, b are relatively prime positive integers, and $ab = c^n$, with n, c positive integers, show that there exists positive integers d, e such that $a = d^n$ and $b = e^n$.

   Solution: The inelegant solution of using unique factorization into primes is acceptable.

2) Find all solutions in positive integers to the Diophantine equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{7}$$

   Solution: Multiply by $7xy$ to obtain

$$7y + 7x = xy,$$

   since 7 divides the LHS, we get that $7|xy$. Since 7 is a prime, it divides either x or y, or both. A case study yields that the only solutions are

$$(x, y) \in \{(8, 56), (14, 14), (56, 8)\}.$$

3) For which positive integers n does it hold that $3\phi(n) = \phi(3n)$, where ϕ denotes the Euler phi-function?

   Solution: Write $n = 3^a m$, where 3 does not divide m. If $a > 0$ then

$$\frac{\phi(3n)}{\phi(n)} = \frac{\phi(3^{a+1})\phi(m)}{\phi(3^a)\phi(m)} = \frac{3^{a+1} - 3^a}{3^a - 3^{a-1}} = 3,$$

   but if $a = 0$ then

$$\frac{\phi(3n)}{\phi(n)} = \frac{\phi(3)\phi(n)}{\phi(n)} = 2.$$

4) Calculate $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)$ for all positive integers n; μ is the Möbius function.

   Solution: Given four consecutive integers, exactly one is divisible by four. That integer is not square-free, and its Möbius value is zero.

5) Let $p \geq 7$ be a prime. Show that there exist a positive quadratic residue n of p such that $n+1$ is a quadratic residue of p as well.

Solution: We have that $\left(\frac{2}{p}\right)\left(\frac{5}{p}\right)\left(\frac{10}{p}\right) = \left(\frac{100}{p}\right) = \left(\frac{10}{p}\right)^2 = 1$. Thus at least one of $2, 5, 10$ is a quadratic residue mod p. $1, 4, 9$ are q.r., as well. Hence, there are $n, n+1$ q.r. with $n \leq 9$. We have used that $p \notin \{2, 5\}$.

6) Given that

$$\sqrt{17} = [4; \overline{8}] = 4 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{\ddots}}},$$

find the continued fraction expansion of $\frac{1}{\sqrt{17}}$ and $-\frac{1}{\sqrt{17}}$.

Solution: Clearly,

$$\frac{1}{\sqrt{17}} = \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{\ddots}}}} = [0; 4, \overline{8}]$$

It is somewhat tricker to show that

$$-\frac{1}{\sqrt{17}} = \cfrac{-1}{4 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{\ddots}}}} = -1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{8 + \cfrac{1}{\ddots}}}} = [-1; 1, 3, \overline{8}].$$

7) Show that the integer $m > 2$ has a primitive root if and only if the congruence $x^2 \equiv 1 \mod m$ has precisely the solutions $x \equiv \pm 1 \mod m$.

Hint: Recall that if $k \geq 3$ is an integer, then 5 has (multiplicative) order 2^{k-2} modulo 2^k.

Solution: This is an exercise from the textbook.