Number theory, Talteori 6hp, Kurskod TATA54, Provkod TEN1
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Each problem is worth 3 points. To receive full points, a solution needs to be complete. Indicate which theorems from the textbook that you have used, and include all auxillary calculations.

You may use the following tools:

- pen and paper
- your textbook
- a dumb calculator
- your telephone, but only for calling the examiner and ask for clarification on the exercises

In particular, you may not use a computer.

8p to pass, 10p for grade 4, 12p for grade 5.

1) Find all integers n such that $n+1$ is *not* divisible by 3 and $n+2$ is divisible by 5.

2) Let n be a positive integer. How many solutions are there to the congruence $x^3 + x \equiv 0 \mod 2^n$?

3) How many primitive roots are there mod 7? Find them all. For each primitive root a mod 7 that you find, check which of the "lifts"

$$a + 7t, \qquad 0 \leq t \leq 6$$

are primitive roots mod 49.

4) Determine the (periodic) continued fraction expansion of $\sqrt{3}$ by finding the minimal algebraic relation satisfied by $\sqrt{3} - 1$.

5) For a positive integer n, let

$$[n] = \{1, 2, \ldots, n\}$$
$$[n]^2 = \{ (i,j) | i, j \in [n] \}$$
$$C(n) = \left\{ (i,j) \in [n]^2 \middle| \gcd(i,j) = 1 \right\}$$

Show that

$$\#C(n) = \sum_{d=1}^{n} \mu(d) \lfloor \frac{n}{d} \rfloor^2.$$

Here $\#S$ is the cardinality of S, μ is the Möbius function, and $\lfloor x \rfloor$ is the fractional part of x, i.e., the largest integer n such that $n \leq x$.