Talteori 6hp, Kurskod TATA54, Provkod TEN1
Juni 03, 2021
LINKÖPINGS UNIVERSITET
Matematiska Institutionen
Examinator: Jan Snellman

Every exercise is worth 3 points, for which a complete solution is needed. 8p is enough for grade 3, 11p for grade 4, 14p for grade 5.

1) Låt T be a right triangle with integer side lengths (including the hypotenuse). Show that the area of T is an integer.

2) Let x have the continued fraction expansion $[1; \overline{2, 3}]$. Determine x.

3) Let $2 < p < q$ be prime, and assume that the integer a is relatively prime to p and to q.

   (a) If $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right) = 1$, what can be said about the solubility of the congruence $x^2 \equiv a \mod pq$?

   (b) What if $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right) = -1$?

   (c) What if $\left(\frac{a}{p}\right) \neq \left(\frac{a}{q}\right)$?

4) Let $f(x) = x^4 - 1$.

   (a) List all zeroes of $f(x)$ in $\mathbf{Z}_{125}$

   (b) List all zeroes of $f(x)$ in $\mathbf{Z}_{49}$

   (c) For $n > 2$, give a sharp lower bound of the number of zeroes of $f(x)$ in $\mathbf{Z}_n$.

5) Find all pairs (x, y), with x, y Gaussian integers, that are soultions to the linear Diophantine equation

$$(2 + i)x + (1 + i)y = i$$

6) The following table show that 2 is a primitive root modulo 29.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$2^k \mod 29$	1	2	4	8	16	3	6	12	24	19	9	18	7	14	28

k	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$2^k \mod 29$	27	25	21	13	26	23	17	5	10	20	11	22	15	1

Now solve

$$7^x \equiv -5 \mod 29$$

7) Show that for each positive integer n it holds that

$$\mu(n)^2 = \sum_{d|n} \mu(d) 2^{\omega(n/d)}$$

where $\omega(k)$ denotes the number of distinct prime factors of k.