Kedjebråk

n pos heltal

$x_0, x_1, \ldots, x_n \in D$, domän

$$S = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots \; x_{n-1} + \cfrac{1}{x_n}}}} = [x_0; x_1, x_2, x_3, \ldots, x_{n-1}, x_n]$$

Ex $[2; 3, 4] = 2 + \cfrac{1}{3 + \frac{1}{4}} = 2 + \cfrac{1}{\frac{13}{4}} = 2 + \frac{4}{13} = \frac{30}{13}$

Varje rationell tal kan skrivas som kedjebråk

Ex: $\frac{93}{33} = \frac{2\cdot 33+30}{33} = 2 + \frac{30}{33} = 2 + \frac{1}{\frac{33}{30}} = 2 + \frac{1}{\frac{1\cdot 30+3}{30}}$

$$= 2 + \frac{1}{1 + \frac{3}{30}} = 2 + \frac{1}{1+\frac{1}{\frac{30}{3}}} = 2 + \frac{1}{1+\frac{1}{10}}$$

$$= [2; 1, 10]$$

jfr: Euklides alg, sgd(93, 33) = 3

$93 = 2 \cdot 33 + 30$

$33 = 1 \cdot 30 + 3$

$30 = 10 \cdot 3 + 0$

Ex

$$\frac{98}{255} = 0 + \frac{1}{\frac{255}{98}} = 0 + \frac{1}{\frac{2\cdot 98 + 59}{98}} = 0 + \frac{1}{2 + \frac{59}{98}}$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{\frac{98}{59}}} = 0 + \cfrac{1}{2 + \cfrac{1}{\frac{1\cdot 59 + 39}{59}}} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{39}{59}}}$$

$$= \left[0; 2, \frac{98}{59}\right]$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{59}{39}}}} = \left[0; 2, 1, \frac{59}{39}\right] = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{1\cdot 39 + 20}{39}}}}$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{20}{39}}}} = \left[0; 2, 1, 1, \frac{39}{20}\right] = \left[0; 2, 1, 1, 1 + \frac{19}{20}\right] = \left[0; 2, 1, 1, 1 + \frac{1}{\frac{20}{19}}\right]$$

$$= \left[0; 2, 1, 1, 1, \frac{20}{19}\right] = \dots = [0; 2, 1, 1, 1, 1, 19]$$

Beräkning baklänges

$$[x_0; x_1] = x_0 + \frac{1}{x_1} = \frac{x_0 x_1 + 1}{x_1} = \left[x_0 + \frac{1}{x_1}\right]$$

$$[x_0; x_1, x_2] = x_0 + \frac{1}{x_1 + \frac{1}{x_2}} = \left[x_0; x_1 + \frac{1}{x_2}\right]$$

$$[1; 2, 3] = \left[1; 2 + \frac{1}{3}\right]$$

$$1 + \frac{1}{2 + \frac{1}{3}} = 1 + \frac{1}{7/3} = 1 + \frac{3}{7} = \frac{10}{7}$$

$$[x_0; x_1, x_2, x_3] = x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3}}} = \left[x_0; x_1, x_2 + \frac{1}{x_3}\right]$$

$$[x_0; x_1, x_2, \ldots, x_{n-1}, x_n] = x_0 + \frac{1}{\ddots\, x_{n-2} + \frac{1}{x_{n-1} + \frac{1}{x_n}}} = \left[x_0; x_1, \ldots, x_{n-2}, x_{n-1} + \frac{1}{x_n}\right]$$

Fundamental relation

Beräkning framlänges --- rekursiv relation mellan konvergenter

$$S = [x_0; x_1, x_2, x_3, x_4] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4}}}}$$

- 0:e konvergent $= [x_0] = \frac{x_0}{1} = \frac{h_0}{k_0}$
- 1:a konvergent $= [x_0; x_1] = x_0 + \frac{1}{x_1} = \frac{x_0 x_1 + 1}{x_1} = \frac{h_1}{k_1}$
- 2:a konvergent $= [x_0; x_1, x_2] = [x_0; x_1 + \frac{1}{x_2}] = \frac{x_0 x_1 x_2 + x_0 + x_2}{x_1 x_2 + 1} = \frac{h_2}{k_2}$
- 3:e: $= [x_0; x_1, x_2, x_3] = [x_0; x_1, x_2 + \frac{1}{x_3}] =$

$$= \frac{x_0x_1x_2x_3 + x_0x_1 + x_0x_3 + x_2x_3 + 1}{x_1x_2x_3 + x_1 + x_3} = \frac{h_3}{k_3}$$

4:e $= [x_0; x_1, x_2, x_3, x_4] = [x_0; x_1, x_2, x_3 + \frac{1}{x_4}] = \frac{x_0x_1x_2x_3x_4 + x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_2x_3x_4 + x_0 + x_2 + x_4}{x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1}$

$$= \frac{h_4}{k_4}$$

$$\frac{h_2}{k_2} = \frac{x_0 x_1 x_2 + x_0 + x_2}{x_1 x_2 + 1} = \frac{x_2(x_0 x_1 + 1) + x_0}{x_2(x_1) + 1} = \frac{x_2 h_1 + h_0}{x_2 k_1 + k_0}$$

$$\frac{h_3}{k_3} = \frac{x_0x_1x_2x_3 + x_0x_1 + x_0x_3 + x_2x_3 + 1}{x_1x_2x_3 + x_1 + x_3} = \frac{x_3 h_2 + h_1}{x_3 k_2 + k_1}$$

$$\frac{h_4}{k_4} = \frac{x_0x_1x_2x_3x_4 + x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_2x_3x_4 + x_0 + x_2 + x_4}{x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1}$$

$$= \frac{x_4 h_3 + h_2}{x_4 k_3 + k_2}$$

Sats $[x_0; x_1, \dots, x_\ell] = \dfrac{h_\ell}{k_\ell}$

med

1) $h_0 = x_0, \quad k_0 = 1$

2) $h_1 = x_0 x_1 + 1, \quad k_1 = x_1$

3) $h_\ell = x_\ell h_{\ell-1} + h_{\ell-2}$

$\quad k_\ell = x_\ell k_{\ell-1} + k_{\ell-2} \qquad \ell \geq 2$

$h_{-1} = 1, \; k_{-1} = 0$

$h_{-2} = 0, \; k_{-2} = 1$

B]

$$\frac{h_\ell}{k_\ell} = [x_0; x_1, x_2, \dots, x_\ell] \overset{\text{Ind. rel}}{=} \left[x_0; x_1, \dots, x_{\ell-2}, x_{\ell-1} + \frac{1}{x_\ell}\right]$$

$$\overset{\text{ind. ant}}{=} \frac{\left(x_{\ell-1} + \frac{1}{x_\ell}\right) h_{\ell-2} + h_{\ell-3}}{\left(x_{\ell-1} + \frac{1}{x_\ell}\right) k_{\ell-2} + k_{\ell-3}} = \frac{x_\ell \left(x_{\ell-1} h_{\ell-2} + h_{\ell-3}\right) + h_{\ell-2}}{x_\ell \left(x_{\ell-1} k_{\ell-2} + k_{\ell-3}\right) + k_{\ell-2}} = \frac{x_\ell h_{\ell-1} + h_{\ell-2}}{x_\ell k_{\ell-1} + k_{\ell-2}} \quad \square$$

$$\begin{bmatrix} h_{-2} \\ k_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} h_{-1} \\ k_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} h_\ell \\ k_\ell \end{bmatrix} = \begin{bmatrix} x_\ell h_{\ell-1} + h_{\ell-2} \\ x_\ell k_{\ell-1} + k_{\ell-2} \end{bmatrix} = x_\ell \begin{bmatrix} h_{\ell-1} \\ k_{\ell-1} \end{bmatrix} + 1 \cdot \begin{bmatrix} h_{\ell-2} \\ k_{\ell-2} \end{bmatrix}$$

Ex Beräkna $[1;2,3,4,5]$

$$\begin{bmatrix} h_{-2} \\ k_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} h_{-1} \\ k_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ k_0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \frac{1}{1} = 1$$

$$\begin{bmatrix} h_1 \\ k_1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \frac{3}{2} = 1,5$$

$$\begin{bmatrix} h_2 \\ k_2 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \qquad \frac{10}{7} \approx 1,429$$

$$\begin{bmatrix} h_3 \\ k_3 \end{bmatrix} = 4 \begin{bmatrix} 10 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 43 \\ 30 \end{bmatrix} \qquad \frac{43}{30} \approx 1,4333$$

$$\begin{bmatrix} h_4 \\ k_4 \end{bmatrix} = 5 \begin{bmatrix} 43 \\ 30 \end{bmatrix} + \begin{bmatrix} 10 \\ 7 \end{bmatrix} = \begin{bmatrix} 255 \\ 157 \end{bmatrix} \qquad \frac{255}{157} \approx 1.4331$$

$$C_\ell = \frac{h_\ell}{k_\ell}, \qquad \begin{bmatrix} h_\ell \\ k_\ell \end{bmatrix} = x_\ell \begin{bmatrix} h_{\ell-1} \\ k_{\ell-1} \end{bmatrix} + \begin{bmatrix} h_{\ell-2} \\ k_{\ell-2} \end{bmatrix}$$

$$C_\ell = \frac{h_\ell}{k_\ell} = \frac{x_\ell h_{\ell-1} + h_{\ell-2}}{x_\ell k_{\ell-1} + k_{\ell-2}} = \frac{h_{\ell-1} + \frac{h_{\ell-2}}{x_\ell}}{k_{\ell-1} + \frac{k_{\ell-2}}{x_\ell}} = \frac{h_{\ell-1}}{k_{\ell-1}} \cdot \frac{\left(1 + \frac{h_{\ell-2}}{h_{\ell-1} \cdot x_\ell}\right)}{\left(1 + \frac{k_{\ell-2}}{k_{\ell-1} \cdot x_\ell}\right)}$$

$$= C_{\ell-1} \cdot \frac{x_\ell h_{\ell-1} \cdot k_{\ell-1} + k_{\ell-1} h_{\ell-2}}{x_\ell h_{\ell-1} k_{\ell-1} + k_{\ell-2} \cdot h_{\ell-1}}$$

Vad är $k_{\ell-1} h_{\ell-2} - k_{\ell-2} h_{\ell-1} = \begin{vmatrix} h_{\ell-2} & h_{\ell-1} \\ k_{\ell-2} & k_{\ell-1} \end{vmatrix}$?

Sats: $k_n h_{n-1} - k_{n-1} h_n = (-1)^n$

B] $VL = (x_n k_{n-1} + k_{n-2}) h_{n-1} - k_{n-1}(x_n h_{n-1} + h_{n-2})$

$= x_n \cancel{k_{n-1}} h_{n-1} + k_{n-2} h_{n-1} - \cancel{k_{n-1} x_n} h_{n-1} - k_{n-1} h_{n-2}$

$= -(k_{n-1} h_{n-2} - k_{n-2} h_{n-1}) \overset{\text{ind.ant}}{=} -(-1)^{n-1} = (-1)^n$ $\square$

Följd $sgd(h_n, k_n) = 1$.

B] $d = sgd(h_n, k_n)$, $d \mid h_n$, $d \mid k_n \Rightarrow d \mid (k_n h_{n-1} - k_{n-1} h_n) \Rightarrow d \mid (-1)^n$

Följd: $\dfrac{h_n}{k_n} - \dfrac{h_{n-1}}{k_{n-1}} = \dfrac{h_n k_{n-1} - k_n h_{n-1}}{k_n k_{n-1}} = \dfrac{(-1)^{n+1}}{k_n k_{n-1}}$

Följd Om alla $x_i \in \mathbb{Z}_+$ så

$x_0 < x_2 < x_4 < \dots x_n \dots < x_7 < x_5 < x_3 < x_1$