Number Theory, Lecture 10

Pell's equation

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Summary

1. **Pell's equation**
   - Variations
   - Trivial cases
   - Relation to CF
   - New solutions from old
2. **Applications**

Definition

- Pell's equation is the Diophantine equation in x, y

$$x^2 - dy^2 = 1$$

  with d an integer
- Negative Pell is

$$x^2 - dy^2 = -1$$

- We also study the Pell-like equations

$$x^2 - dy^2 = n$$

  where n is an integer

Trivial cases

Study

$$x^2 - dy^2 = n$$

- If $d, n < 0$ then no solution
- Id $d < 0$, $n > 0$ then a solution satisfies $|x| \leq \sqrt{n}$, $|y| \leq \sqrt{n/|d|}$, so finitely many solns
- if $d = D^2$ then

  $$n = x^2 - dy^2 = x^2 - D^2y^2 = (x + Dy)(x - Dy)$$

  so soln correspond to soln to eqn sys

  $$x + Dy = a$$
  $$x - Dy = b$$
  $$ab = n$$

  and again ,finitely many solns

Theorem

Suppose $0 < d$, $|n| < \sqrt{d}$, d *not a square. If* $(x, y) \in \mathbb{Z}^2$ *satisfies* $x^2 - dy^2 = n$, *then* x/y *is a convergent of the CF of* $\sqrt{d}$.

Proof.

Assume $n > 0$, then

$$(x + y\sqrt{d})(x - y\sqrt{d}) = n,$$

so $x - y\sqrt{d} > 0$, so $x > y\sqrt{d}$, so $\frac{x}{y} - \sqrt{d} > 0$. Then

$$\frac{x}{y} - \sqrt{d} = \frac{x - \sqrt{d}y}{y} = \frac{x^2 - dy^2}{y(x + y\sqrt{d})} < \frac{|n|}{y(2y\sqrt{d})} < \frac{\sqrt{d}}{2y^2\sqrt{d}} = \frac{1}{2y^2}$$

Such good approximation must come frome a convergent. □

CF of $\sqrt{d}$

Theorem

d positive integer, not square. Then the CF of $\sqrt{d} = [a_0, a_1, a_2, \ldots]$, and the corresponding convergents p_k/q_k, can be computed as follows:

1. $\alpha_0 = \sqrt{d}$, $a_0 = \lfloor \alpha_0 \rfloor$, $P_0 = 0$, $Q_0 = 1$, $p_0 = a_0, q_0 = 1$
2. $\alpha_k = \frac{P_k + \sqrt{d}}{Q_k}$, $a_k = \lfloor \alpha_k \rfloor$
3. $P_{k+1} = a_k Q_k - P_k$, $Q_{k+1} = (d - P_{k+1}^2)/Q_k$
4. $$P_{k+1} p_k - n q_k = -Q_{k+1} p_{k-1}$$ $$p_k - P_{k+1} q_k = Q_{k+1} q_{k-1}$$

For all k,

$$p_k^2 - d q_k^2 = (-1)^{k+1} Q_{k+1}$$

Theorem

d positive integer, not a square. Let $\sqrt{d} = [a_0, a_1, \ldots]$, and let n be the period length of this periodic CF expansion. Let p_k/q_k be the k'th convergent.

- *If n even, negative Pell has no solns, and Pell $x^2 - dy^2 = 1$ has precisely the solns $x = p_{jn-1}$, $y = q_{jn-1}$, $j = 1, 2, 3 \ldots$*
- *If n odd, negative Pell has precisely the solns $x = p_{(2j-1)n-1}$, $y = q_{(2j-1)n-1}$, $j = 1, 2, 3, \ldots$, and Pell has precisely the solns $x = p_{2jn-1}$, $y = q_{2jn-1}$, $j = 1, 2, 3, \ldots$.*

Proof.

See Rosen. □

Example

$\sqrt{17} = [4, \overline{8}]$, so the period length is 1. The odd numbered convergents are

$$33/8, 2177/528, 143649/34840, 9478657/2298912, \ldots$$

and indeed $33^2 - 17 * 8^2 = 1$. The even numbered convergents are

$$268/65, 17684/4289, 1166876/283009, 76996132/18674305, \ldots$$

and indeed $268^2 - 17 * 65^2 = -1$.

Lemma

$$(x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2$$

so if (x_1, x_2), (x_2, y_2) *are solns to (standard) Pell, then so is* $(x_1x_2 + dy_1y_2, x_1y_2 + x_2y_1)$.
In particular, $(x_1^2 + dy_1^2, 2x_1y_1)$, *is a solution.*

Proof.

Obvious. □

Note that

$$(x + \sqrt{d}y)^2 = x^2 + dy^2 + \sqrt{d}2xy.$$

Theorem

1. *If* (x_1, y_1) *is a soln to* $x^2 - dy^2 = 1$, *then writing*

$$(x_1 + y_1\sqrt{d})^k = x_k + \sqrt{d}y_k,$$

   it holds that (x_k, y_k) *is also a soln to (standard) Pell.*

2. *All solns to standard Pell are obtainable from the smallest soln* (x_1, y_1), *by the above procedure.*

Proof.

1. Easy.
2. Hard, see Rosen.

□

Example

We return to

$$x^2 - 17y^2 = 1,$$

with smallest soln $(x_1, y_1) = (33, 8)$ We calculate that

$$(33 + 8\sqrt{17})^2 = 33^2 + 17 * 8^2 + 16 * 33 * \sqrt{17} = 2177 + 528\sqrt{17},$$

so $(2177, 528)$ is the next soln.

Double equations

Example

Eliminating t from the pair of equations

$$x^2 - 21t - 11 = 0$$
$$y^2 - 7t - 9 = 0$$

gives the Pell-type eqn $x^2 - 3y^2 + 16 = 0$.

$$x^2 - 21 * t - 7 = 0$$
$$y^2 - 7 * t - 2 = 0$$

gives $x^2 - 3 * y^2 = 1$.

Approximating square roots

Example

Since $(x, y) = (2177, 528)$ is a soln to $x^2 - 17y^2 = 1$, we have that

$$4.1231 \approx \sqrt{17} = \sqrt{\frac{x^2-1}{y^2}} = \frac{\sqrt{x^2-1}}{y} \approx \frac{x}{y} = \frac{2177}{528} \approx 4.1231$$

Sums of consecutive integers

Problem

When is $\sum_{k=1}^{n} k = \sum_{k=n+1}^{m} k$?

$$
\begin{aligned}
LHS - RHS &= \frac{n(n+1)}{2} - \frac{n+1+m}{2}(m-n) \\
&= \frac{1}{2}\left(n^2+n-nm+n^2-m+n-m^2+mn\right) \\
&= \frac{1}{2}\left(2n^2+2n-m^2-m\right) = \frac{1}{4}(4n^2+4n-2m^2-2m) \\
&= \frac{1}{4}(2((2n+1)^2-1)-((2m+1)^2-1) \\
&= \frac{1}{4}((2m-1)^2-2(2n+1)^2+1) \\
&= \frac{1}{4}(s^2-2t^2+1)
\end{aligned}
$$