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Möbius inversion

Multiplicativity is preserved by multiplication Matrix verification Divisor functions Euler φ again μ itself

Number Theory, Lecture 3 Arithmetical functions, Dirichlet convolution, Multiplicative functions, Möbius inversion

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Definition

An arithmetical function is a function $f : \mathbb{P} \to \mathbb{C}$.

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We will mostly deal with integer-valued a.f. Euler \phi is one:
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Number

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Mobius inversion Multiplicativity is

preserved by multiplication Matrix verification Divisor functions Euler φ again $n = p_1^{a_1} \cdots p_r^{a_r}, \quad q_i \text{ distinct primes}$ Liouville function λ . Möbius function μ :

$$\begin{split} & \omega(n) = r \\ & \Omega(n) = a_1 + \dots + a_r \\ & \lambda(n) = (-1)^{\Omega(n)} \\ & \mu(n) = \begin{cases} \lambda(n) & \omega(n) = \Omega(n) \\ 0 & \text{otherwise} \end{cases} \end{split}$$

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$$d(n) = \sum_{\substack{k|n \\ \sigma(n) = \sum_{\substack{k|n \\ gcd(k,n) = 1}}} 1$$

1

Even more Arithmetical functions

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 $\begin{array}{l} \mbox{Möbius} \\ \mbox{inversion} \\ \mbox{Multiplicativity is} \\ \mbox{preserved by} \\ \mbox{multiplication} \\ \mbox{Matrix verification} \\ \mbox{Matrix verification} \\ \mbox{Divisor functions} \\ \mbox{Euler } \phi \mbox{ again} \\ \mbox{μ itself} \end{array}$

p prime. Von Mangoldt function Λ , prime-counting function π , Legendre symbol $\left(\frac{n}{p}\right)$, p-valuation v_p .

$$\begin{split} \Lambda(n) &= \begin{cases} \log q & n = q^k, \ q \ \text{prime} \\ 0 & \text{otherwise} \end{cases} \\ \pi(n) &= \sum_{\substack{1 \le k \le n \\ k \ \text{prime}}} 1 \\ & \left(\frac{n}{p}\right) = \begin{cases} 0 & n \equiv 0 \mod p \\ +1 & n \not\equiv 0 \mod p \text{ and exists } a \text{ such that } n \equiv a^2 \mod p \\ -1 & n \not\equiv 0 \mod p \text{ and exists no } a \text{ such that } n \equiv a^2 \mod p \end{cases} \\ v_p(n) &= k, \ p^k | n, \ p^{k+1} \not\mid n \end{split}$$

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$$\mathbf{e}(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$
$$\mathbf{0}(n) = 0$$
$$\mathbf{1}(n) = 1 \qquad \text{often denoted by } \zeta$$
$$\mathbf{I}(n) = n$$
$$\mathbf{e}_i(n) = \begin{cases} 1 & n = i \\ 0 & n \neq i \end{cases}$$

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Definition

Let f, g be arithmetical functions. Then their *Dirichlet convolution* is another a.f., defined by

$$(f * g)(n) = \sum_{\substack{1 \le a, b \le n \\ ab = n}} f(a)g(b) = \sum_{\substack{1 \le k \le n \\ k \mid n}} f(k)g(n/k) = \sum_{\substack{1 \le \ell \le n \\ \ell \mid n}} f(n/\ell)g(\ell)$$
(DC)

Example

$$(f * g)(10) = f(1)g(10) + f(2)g(5) + f(5)g(2) + f(10)g(1)$$

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- f * (g * h) = (f * g) * h
- f * g = g * f
- There is a unit for this multiplication, $\mathbf{e}(1)=1$, $\mathbf{e}(n)=0$ for n>1
- Not all a.f. are invertible
- We can add: (f + g)(n) = f(n) + g(n)
- We can scale: (cf)(n) = cf(n)
- **0**(*n*) = 0 is a zero vector
- A \mathbb{C} -vector space with multiplication; an *algebra*.

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- Let $n \in \mathbb{P}$ and $D(n) = \{1 \le k \le n | k | n\}$ be its divisors
- We want to understand a.f. restricted to D(n), in particular their multiplication
- Given a.f. f, form matrix A with rows and columns indexed by elems in D(n), and A_{ij} = f(j/i) if i|j, 0 otherwise
- Similarly for a.f. g and matrix B
- Then AB is the matrix for f * g

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Example

• n = 12, D(n) as follows



- *f* = 1
- *A* = **??**



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- $F(n) = (1 * f)(n) = \sum_{k|n} f(k)$
- The summation of f
- Sometimes F is known and we want to recover f
- .

$$F(1) = f(1)$$

$$F(2) = f(1) + f(2)$$

$$F(3) = f(1) + f(3)$$

$$F(4) = f(1) + f(2) + f(4)$$

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Theorem

Proof.

f has inverse $g = f^{-1}$ iff $f(1) \neq 0$

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$$1 = (f * g)(1) = f(1)g(1)$$

$$0 = (f * g)(2) = f(1)g(2) + f(2)g(1)$$

$$0 = (f * g)(3) = f(1)g(3) + f(3)g(1)$$

$$0 = (f * g)(4) = f(1)g(4) + f(2)g(2) + f(4)g(1)$$

$$0 = (f * g)(5) = f(1)g(5) + f(5)g(1)$$

$$\vdots$$

$$0 = (f * g)(n) = f(1)g(n) + \sum_{\substack{k|n\\1 \le k \le n}} f(k)g(n/k)$$

Want f * g = e, so (f * g)(m) = 1 if m = 1, 0 otherwise. Gives

so, by induction, we can solve for g(n).

Inverses

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Definition

If $f \neq \mathbf{0}$, then the *order* of f is

 $\operatorname{ord}(f) = \min\{n | f(n) \neq 0\}$

and the norm

$$\|f\|=2^{-\mathrm{ord}(f)}$$

Lemma

- $f = \sum_{n} f(n)e_{n}$, i.e., the partial sums of this sum converge to f
- *if* f(1) = 0 *then* e + f *is invertible, with inverse given by convergent geometric series:*

$$\frac{e}{e+f} = e-f+f*f-f*f+\cdots$$

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Definition

- f is totally multiplicative if f(nm) = f(n)f(m)
- f is multiplicative if f(nm) = f(n)f(m) whenever gcd(n, m) = 1

Theorem

- Let $n = \prod_{i} p_{i}^{a_{i}}$, prime factorization. Then
 - If f mult then $f(n) = \prod_j f(p^j)$, i.e., f is determined by its values at prime powers
 - If f tot mult then $f(n) = \prod_{i} f(p)^{j}$, i.e., f is determined by its values at primes

Proof.

Obvious!

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Theorem

The Euler ϕ function is multiplicative.

Proof

Let gcd(m, n) = 1. Want to prove $\phi(mn) = \phi(m)\phi(n)$, in other words,

$$|\mathbb{Z}_{mn}| = |\mathbb{Z}_m| \, |\mathbb{Z}_n| \tag{1}$$

(2)

Claim: following bijection:

$$\mathbb{Z}_{mn} \ni [a]_{mn} \mapsto ([a]_m, [a]_n) \in \mathbb{Z}_m \times \mathbb{Z}_n$$

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Proof.

- Well-defined, since a ≡ a' mod mn implies a ≡ a' mod m and a ≡ a' mod n.
- Injective, since $a \equiv a' \mod m$ and $a \equiv a' \mod n$ implies $a \equiv a' \mod mn$
- Surjective, by the CRT: take c, d, then exists x with

$$x \equiv c \mod m$$
$$x \equiv d \mod n$$

so $[x]_{mn} \mapsto ([c]_m, [d]_n)$

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1 Take *p* prime

6 So

2 Then all $1 \le a \le p$ relatively prime to p, so $\phi(p) = p - 1$

(3) Now consider prime power p^r

4 For $1 \le a \le p^r$, $gcd(a, p^r) > 1$ iff p|n

5 Example:
$$p = 3$$
, $r = 2$:
6 So $\phi(p^r) = p^r - \frac{p^r}{p} = p^r \left(1 - \frac{1}{p}\right)$
7 For $n = p_1^{r_1} \cdots p_s^{r_s}$, we have by multiplicativity

 $\Phi(p_1^{r_1}\cdots p_s^{r_s}) = \Phi(p_1^{r_1})\cdots \Phi(p_s^{r_s})$ $= p_1^{r_1} \cdots p_c^{r_s} (1 - 1/p_1) \cdots (1 - 1/p_s)$ $= n \prod (1 - 1/p_i)$

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Example

• $\phi(15) = \phi(3)\phi(5) = 2 * 4 = 8$

•
$$\phi(16) = \phi(2^4) = 2^4 - 2^3 = 8$$

• $\phi(120) = \phi(2^3 * 3 * 5) = 120(1 - 1/2)(1 - 1/3)(1 - 1/5) = 120 * (4/15) = 32.$

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n = p gives $\phi(n) = n - 1$. This is visible in graph of $\phi(n)$.



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f, g (non-zero) multiplicative arithmetical functions, h = f * g

(i) e is multiplicative (ii) f(1) = 1, so f is invertible (iii) f^{-1} is multiplicative

Proof

Theorem

(i-ii) Trivial. (iii): Suppose gcd(m, n) = 1. Then

$$h(mn) = (f * g)(mn) = \sum_{k|mn} f(k)g(\frac{mn}{k}) = \sum_{\substack{k_1|m \\ k_2|n}} f(k_1k_2)g(\frac{m}{k_1}\frac{n}{k_2})$$
$$= \sum_{\substack{k_1|m \\ k_2|n}} f(k_1)f(k_2)g(\frac{m}{k_1})g(\frac{n}{k_2}) = \sum_{\substack{k_1|m \\ k_2|n}} f(k_1)g(\frac{m}{k_1})\sum_{\substack{k_2|n \\ k_2|n}} f(k_2)g(\frac{n}{k_2}) = h(m)h(n)$$

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Proof.

(iv): The formula for the inverse now becomes

 $f^{-1}(nm) = -\sum_{m} f^{-1}(d)f(\frac{nm}{d})$

d|n

d<n

$$f^{-1}(n) = -\sum_{\substack{d \mid n \\ d < n}} f^{-1}(d) f(\frac{nm}{d})$$

=

 $-\sum f^{-1}(d_1d_2)f(\frac{nm}{d_1d_2})$

 $d_1|n$

 $d_2 \mid m$ $d_1 d_2 < n$

so if
$$\gcd({\it n},{\it m})=1$$
 then

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Assume, by induction that f^{-1} is multiplicative for arguments < nm.

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Theorem (Möbius inversion)

$1 * \mu = e$

2
$$F(n) = \sum_{k|n} f(k)$$
 for all n iff $f(n) = \sum_{k|n} F(k)\mu(n/k)$ for all n

Proof.

(1): Since the a.f. involved are multiplicative (check!), it suffices to check on prime powers p^r . Then $(1 * \mu)(p^0) = 1$, and for r > 0

$$(\mu * \mathbf{1})(p^r) = \sum_{k=0}^r \mu(p^k) = 1 - 1 + 0 + \dots + 0 = 0.$$

(2): If
$$F = f * 1$$
 then $f = f * e = f * 1 * \mu = F * \mu$.

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Example

• n = 12, D(n) as follows

- f = 1
- *A* = ??
- $g = \mu$
- C = ??
- AC = ??

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Recall

$$d(n) = \sum_{k|n} 1, \qquad \sigma(n) = \sum_{k|n} k$$

We can write this as

 $d = 1 * 1, \qquad \sigma = 1 * I$

from which we conclude that d, σ are multiplicative, and that

$$\mu * d = 1, \qquad \mu * \sigma = \mathbf{I}$$

or in other words

$$\sum_{k|n} \mu(k)d(n/k) = 1, \qquad \sum_{k|n} \mu(k)\sigma(n/k) = n$$

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Definition

$$\sigma_{k}(n) = \sum_{d \mid n} d^{k}$$
. In particular, $\sigma_{0} = d$, $\sigma_{1} = \sigma_{2}$

Lemma

σ

σ_k is multiplicative

Proof.

Suppose gcd(m, n) = 1. Then

$$\sigma_k(mn) = \sum_{d|mn} d^k = \sum_{\substack{d_1|m\\d_2|n}} (d_1d_2)^k = \sum_{\substack{d_1|m\\d_2|n}} d_1^k d_2^k = \sum_{\substack{d_1|m\\d_2|n}} d_1^k \sum_{\substack{d_2|n\\d_2|n}} d_2^k = \sigma_k(m)\sigma_k(n)$$

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Theorem

1
$$\sigma_k(p_1^{a_1}\cdots p_r^{a_r}) = \prod_{j=1}^r \frac{1-p_j^{k(a_j+1)}}{1-p_j^k}$$

2 $\sum_{d|n} d^k \mu(n/d) = n^k$

Proof.

Try to prove it yourself!

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Lemma

 $\mathbf{1} \ast \varphi = \mathbf{I}$

Proof.

In other words, want prove

$$\sum_{k|n} \phi(k) = n.$$

```
Multiplicative, so put n = p^r.

If r = 0: LHS = 1, OK.

If r > 0: LHS = \sum_{j=0}^{r} \phi(p^j) = 1 + \sum_{j=1}^{r} (p^j - p^{j-1}) = p^r, since sum telescoping.
```

Divisors of 12

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$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(6) + \varphi(12) = 1 + 1 + 2 + 2 + 2 + 4 = 12$$



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$$\phi(n) = \sum_{k|n} \mu(k) \frac{n}{k} = \sum_{k|n} k \mu(\frac{n}{k})$$

Proof.

Since

 $\mathbf{1} * \mathbf{\phi} = \mathbf{I},$

we have that

 $\phi = \mu * \mathbf{I} = \mathbf{I} * \mu$

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Multiplicativity is

ellman

An *n*'th root of unity is a complex root to $z^n = 1$. A primitive *n*'th root of unity is not a *k*'th root of unity for smaller *k*.

Lemma

Definition

Put $\xi_n = \exp(\frac{2\pi}{n}i)$. Then the n'th roots of unity are ξ_n^s , $1 \le s \le n$, and the primitive n'th roots of unity are ξ_n^k , gcd(k, n) = 1.

Lemma

If n > 1,

$$\sum_{s=1}^{n} \xi_{n}^{s} = \frac{\xi_{n}^{n} - 1}{\xi_{n} - 1} = 0.$$

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Euler of aga





Let f(d) denote the sum of the primitive d'th roots of unity. Then f(1) = 1, and for n > 1, $\sum_{d|n} f(d) = 0$. So 1 * f = e, hence $f = \mu$. So the Möbius function is the sum of the primitive roots.