Number Theory, Lecture 7

Continued fractions

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Summary

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Example

$$5+\frac{1}{3+\frac{1}{11+\frac{1}{2}}}=5+\frac{1}{3+\frac{2}{23}}=5+\frac{23}{71}=\frac{378}{71}$$

Example

$$5 = 5 \approx 5.000$$

$$5 + \frac{1}{3} = \frac{16}{3} \approx 5.333$$

$$5 + \cfrac{1}{3 + \cfrac{1}{11}} = \frac{181}{34} \approx 5.323$$

$$5 + \cfrac{1}{3 + \cfrac{1}{11 + \cfrac{1}{2}}} = \frac{378}{71} \approx 5.324$$

Example

$$\frac{720}{164} = 4 + \frac{64}{164} = 4 + \frac{1}{\frac{164}{64}} = 4 + \frac{1}{2 + \frac{36}{64}}$$

$$= 4 + \frac{1}{2 + \frac{1}{\frac{64}{36}}} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{28}{36}}} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{36}{28}}}} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{8}{28}}}}$$

$$= 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{28}{8}}}}} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{4}{8}}}}}$$

$$= 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{\frac{8}{4}}}}}} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{2}}}}}$$

Example (Cont)

Notation:

$$4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\frac{1}{2}}}}} = [4,2,1,1,3,2]$$

Convergents:

$$[4,] = 4, \qquad [4,2] = 4+\frac{1}{2} = \frac{9}{2}$$

$$[4,2,1] = 4+\frac{1}{2+\frac{1}{1}} = \frac{13}{3}, \qquad [4,2,1,1] = 4+\frac{1}{2+\frac{1}{1+\frac{1}{1}}} = \frac{22}{5}$$

$$[4,2,1,1,3] = 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{3}}}} = \frac{79}{18}$$

$$[4,2,1,1,3,2] = 4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\frac{1}{2}}}}} = \frac{180}{41} = \frac{180*4}{41*4} = \frac{720}{164}$$

Example (Cont)

Compare with Euclides alg:

$$
\begin{aligned}
720 &= 4 * 164 + 64 \\
164 &= 2 * 64 + 36 \\
64 &= 1 * 36 + 28 \\
36 &= 1 * 28 + 8 \\
28 &= 3 * 8 + 4 \\
8 &= 2 * 4
\end{aligned}
$$

$\gcd(720, 164) = 4$, $\frac{720}{164} = \frac{180}{41} = [4, 2, 1, 1, 3, 2]$.

Example

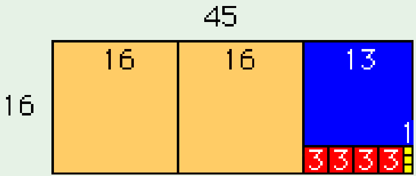
Another example:

$$
\begin{aligned}
45 &= 2 * 16 + 13 \\
16 &= 1 * 13 + 3 \\
13 &= 4 * 3 + 1 \\
3 &= 3 * 1 + 0
\end{aligned}
$$

so

$$\frac{45}{16} = [2, 1, 4, 3] = 2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{3}}}$$

Geometric picture:



Example

- $\frac{31}{18} = [1,1,2,1,1,2] = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$
- $\frac{18}{31} = [0;1,1,2,1,1,2] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}$
- $[2,3,4] = 2 + \frac{1}{3+\frac{1}{4}} = 2 + \frac{1}{3+\frac{1}{3+\frac{1}{1}}} = [2,3,3,1]$

- If $c_0 > 0$ then $1/[c_0, \ldots, c_n] = [0, c_0, \ldots, c_n]$; otherwise $1/[0, c_1, \ldots, c_n] = [c_1, \ldots, c_n]$
- If $m > 1$ then

$$[0, m] = \frac{1}{m} = \frac{1}{m - 1 + \frac{1}{1}} = [0, m-1, 1]$$

and similarly

$$[c_0, \ldots, c_{n-1}, m] = [c_0, \ldots, c_{n-1}, m-1, 1]$$

We prefer the first form

- We also have that $[c_0, \ldots c_{n+1}] = [c_0, \ldots, c_{n-1}, c_n + \frac{1}{c_{n+1}}]$

Theorem

Let a, b be positive integers. Then there is a unique way of writing a/b as a finite continued fraction, such that

$$\frac{a}{b} = [c_0, c_1, c_2, \ldots, c_n]$$

with $c_i \in \mathbb{Z}$, $c_0 \geq 0$, $c_j \geq 1$ for $j \geq 1$, $c_n \geq 2$ if $n > 0$
If $a < b$ then $c_0 = 0$ and

$$(c_1, \ldots, c_n) \mapsto [0, c_1, \ldots, c_n]$$

is a bijection between finite sequences of positive numbers and $\mathbb{Q} \cap [0, 1]$.

Proof.

WLOG $a > b$.

Existence: Euclidean algorithm.

Uniqueness: if $a/b = c_0 + \frac{1}{[c_1,c_2,\ldots,c_n]} = d_0 + \frac{1}{[d_1,d_2,\ldots,d_n]}$, then since $\frac{1}{[c_1,c_2,\ldots,c_n]} < 1$, it follows that $c_0 = \lfloor a/b \rfloor$, and similarly $d_0 = \lfloor a/b \rfloor$. Thus $c_0 = d_0$.

Subtract, and consider

$$\frac{1}{[c_1, c_2, \ldots, c_n]} = \frac{1}{[d_1, d_2, \ldots, d_n]} \quad \Longrightarrow [c_1, c_2, \ldots, c_n] = [d_1, d_2, \ldots, d_n]$$

Then $c_1 = d_1$, and so on. $\square$

Example

We tabulate the continued fraction expansion of $k/13$ for $1 \leq k \leq 12$:

k	CF
1	[0; 13]
2	[0; 6, 2]
3	[0; 4, 3]
4	[0; 3, 4]
5	[0; 2, 1, 1, 2]
6	[0; 2, 6]
7	[0; 1, 1, 6]
8	[0; 1, 1, 1, 1, 2]
9	[0; 1, 2, 4]
10	[0; 1, 3, 3]
11	[0; 1, 5, 2]
12	[0; 1, 12]

Note that the reverse of every CF occurs in the list (possibly written in the non-standrad form).

We can explain this phenomena using the following result:

Lemma

Suppose that $a_0 > 0$, $[a_0, \ldots, a_n] = \frac{A}{B}$, $[a_0, \ldots, a_{n-1}] = \frac{C}{D}$. *Then*

$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{A}{C}$$

Proof.

Needs Euler's rule. □

Example

$$[2, 3, 7] = 2 + \frac{1}{3 + \frac{1}{7}} = \frac{51}{22}$$

$$[2, 3] = 2 + \frac{1}{3} = \frac{7}{3}$$

$$[7, 3, 2] = \frac{51}{7}$$

Rational function

- $[c_0] = c_0,$
- $[c_0, c_1] = c_0 + \frac{1}{c_1} = \frac{c_0 c_1 + 1}{c_1}$
- $[c_0, c_1, c_2] = [c_0, \frac{1}{[c_1, c_2]}] = c_0 + \frac{1}{[c_1, c_2]} = c_0 + \frac{c_2}{c_1 c_2 + 1} = \frac{c_0 c_1 c_2 + c_0 + c_2}{c_1 c_2 + 1}$
- $[c_0, c_1, c_2, c_3] = [c_0, \frac{1}{[c_1, c_2, c_3]}] = c_0 + \frac{1}{[c_1, c_2, c_3]} = c_0 + \frac{c_2 c_3 + 1}{c_1 c_2 c_3 + c_1 + c_3} = \frac{c_0 c_1 c_2 c_3 + c_0 c_1 + c_0 c_3 + c_2 c_3 + 1}{c_1 c_2 c_3 + c_1 + c_3}$
- In general, $[c_0, \ldots c_{n+1}] = [c_0, \ldots, c_{n-1}, c_n + \frac{1}{c_{n+1}}] = c_0 + \frac{1}{[c_1, \ldots, c_{n+1}]} = [c_0, [c_1, \ldots, c_{n+1}]]$
- Thus $[c_0, \ldots, c_n] = \frac{p_n(c_0, \ldots, c_n)}{q_n(c_0, \ldots, c_n)}$ with $p_n, q_n \in \mathbb{Z}[c_0, \ldots, c_n]$

Theorem (Euler's rule)

Let $\mathcal{L}_n(c_0, \ldots, c_n)$ be the polynomial which is the sum of

- *the product $T_n = c_0 c_1 \cdots c_n$,*
- *all factors of T_n obtained by removing a consecutive pair $c_i c_{i+1}$,*
- *all factors of T_n obtained by removing two disjoint consecutive pairs $c_i c_{i+1}$ and $c_\ell c_{\ell+1}$,*
- *all factors obtained by removing three disjoint consecutive pairs, and so on*

Then

- $[c_0, \ldots, c_n] = \frac{\mathcal{L}_n(c_0,\ldots,c_n)}{\mathcal{L}_{n-1}(c_1,\ldots,c_n)}$
- $\mathcal{L}_n(c_0, \ldots, c_n) = c_0 \mathcal{L}_{n-1}(c_1, \ldots, c_n) + \mathcal{L}_{n-2}(c_2, \ldots, c_n)$
- $\mathcal{L}_n(c_0, \ldots, c_n)$ *is invariant under reversal of the order of its variables*
- $\mathcal{L}_n(c_0, \ldots, c_n) = c_n \mathcal{L}_{n-1}(c_0, \ldots, c_{n-1}) + \mathcal{L}_{n-2}(c_0, \ldots, c_{n-2})$

Sketch of proof.

Induction on n. By induction hypothesis,

$$[c_0, \ldots, c_n] = \frac{\mathcal{L}_n(c_0, \ldots, c_n)}{\mathcal{L}_{n-1}(c_1, \ldots, c_n)}$$

Then

$$\begin{aligned}
[c_0, \ldots, c_n, c_{n+1}] &= c_0 + \frac{1}{[c_1, \ldots, c_n]} \\
&= c_0 + \frac{\mathcal{L}_{n-1}(c_1, \ldots, c_n)}{\mathcal{L}_n(c_0, \ldots, c_n)} \\
&= c_0 + \frac{\mathcal{L}_{n-1}(c_1, \ldots, c_n)}{\mathcal{L}_n(c_0, \ldots, c_n)} \\
&= \frac{c_0 \mathcal{L}_n(c_0, \ldots, c_n) + \mathcal{L}_{n-1}(c_1, \ldots, c_n)}{\mathcal{L}_n(c_0, \ldots, c_n)} \\
&= \frac{\mathcal{L}_{n+1}(c_0, \ldots, c_n, c_{n+1})}{\mathcal{L}_n(c_1, \ldots, c_n, c_{n+1})}
\end{aligned}$$

□

Example

- $\mathcal{L}_0(c_0) = c_0$
- $\mathcal{L}_1(c_0, c_1) = c_0 c_1 + 1$
- $\mathcal{L}_2(c_0, c_1, c_2) = c_0 \mathcal{L}_1(c_1, c_2) + \mathcal{L}_0(c_0) = c_0(c_1 c_2 + 1) + c_0 = c_0 c_1 c_2 + c_0 + c_2$
- $[c_0, c_1, c_2] = \frac{c_0 c_1 c_2 + c_0 + c_2}{c_1 c_2 + 1}$
- $[3, 5, 7] = 3 + \frac{1}{5 + \frac{1}{7}} = \frac{3*5*7+3+7}{5*7+1}$

Convergents I

- $[c_0, \ldots, c_n] = \frac{\mathcal{L}_n(c_0,\ldots,c_n)}{\mathcal{L}_{n-1}(c_1,\ldots,c_n)} = \frac{A_n}{B_n}$
- $[c_0] = c_0 = \frac{c_0}{1} = \frac{A_0}{B_0}$
- $[c_0, c_1] = c_0 + \frac{1}{c_1} = \frac{c_0 c_1 + 1}{c_1} = \frac{A_1}{B_1}$
- $\mathcal{L}_n(c_0, \ldots, c_n) = c_n \mathcal{L}_{n-1}(c_0, \ldots, c_{n-1}) + \mathcal{L}_{n-2}(c_0, \ldots, c_{n-2})$
- $A_n = c_n A_{n-1} + A_{n-2}$
- $B_n = c_n B_{n-1} + B_{n-2}$
- $\begin{bmatrix} A_n \\ B_n \end{bmatrix} = c_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix} = \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} c_n \\ 1 \end{bmatrix}$
- $\begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix} = \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix}$
- $\begin{vmatrix} c_n & 1 \\ 1 & 0 \end{vmatrix} = -1$

Convergents II

- $\begin{vmatrix} A_0 & A_1 \\ B_0 & B_1 \end{vmatrix} = \begin{vmatrix} c_0 & c_0 c_1 + 1 \\ 1 & c_1 \end{vmatrix} = -1$
- $A_n B_{n-1} - A_{n-1} B_n = \begin{vmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{vmatrix} = \begin{vmatrix} A_1 & A_0 \\ B_1 & B_0 \end{vmatrix} \begin{vmatrix} c_2 & 1 \\ 1 & 0 \end{vmatrix} \cdots \begin{vmatrix} c_n & 1 \\ 1 & 0 \end{vmatrix} = (-1)^{n-1}$
- Thus, if c_i pos integers, $\gcd(A_i, B_i) = 1$
- Let $\alpha = \alpha_n = [c_0, \ldots, c_n] = \frac{A_n}{B_n}$. Then for $1 \leq i \leq n$, $\alpha_i - \alpha_{i-1} = \frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}} = \frac{(-1)^{i-1}}{B_i B_{i-1}}$
- Furthermore $|\alpha - \alpha_i| < |\alpha - \alpha_{i-1}|$
- Furthermore $\alpha_0 < \alpha_2 < \cdots < \alpha < \cdots < \alpha_3 < \alpha_1$

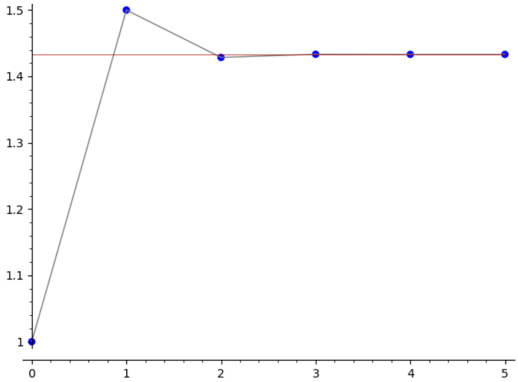
Linear Diophantine eqns

- Study $ax - by = 1$, $\gcd(a, b) = 1$
- Suppose $\frac{a}{b} = [c_0, \ldots, c_n]$
- Last convergent is $\frac{A_n}{B_n} = \frac{a}{b}$
- From $A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}$ we get $aB_{n-1} - A_{n-1} b = (-1)^{n-1}$
- If n odd: $x = B_{n-1}$, $y = A_{n-1}$
- If n even: $[c_0, \ldots, c_n] = [c_0, \ldots, c_n - 1, 1]$, do as above

The rational number $\frac{1393}{972} \approx 1.43312757201646$ has the following convergents:

i	CF	conv	value
0	[1]	1	1.00000000000000
1	[1,2]	3/2	1.50000000000000
2	[1,2,3]	10/7	1.42857142857143
3	[1,2,3,4]	43/30	1.43333333333333
4	[1,2,3,4,5]	225/157	1.43312101910828
5	[1,2,3,4,5,6]	1393/972	1.43312757201646

The convergents converge to the exact value as follows:



Example

We want to solve $474x - 151y = 1$. The CF is

$$[3,7,5,4] = \frac{474}{151},$$

and the second to last convergent is

$$[3,7,5] = \frac{113}{36}$$

Since $n = 3$ is odd, $x = 36$, $y = 113$ should be a solution of the linear Diophantine equation. Indeed,

$$474 * 36 - 151 * 113 = 1$$

Example

We want to solve $113x - 36y = 1$. The CF is

$$[3, 7, 5] = \frac{113}{36}$$

and the penultimate convergent is

$$[3, 7] = \frac{22}{7}$$

Since $n = 2$ is even, $x = 7$, $y = 22$ should be a solution of the linear Diophantine equation with RHS -1. Indeed,

$$113 * 7 - 36 * 22 = -1$$

Writing

$$[3, 7, 5] = [3, 7, 4, 1], \quad [3, 7, 4] = \frac{91}{29}$$

gives $x = 29$, $y = 91$, which solve the original LDE.

Finite continued fractions, repetition

- The following treatment is almost verbatim from **Stein**, *Elementary Number Theory*
- Assume $a_0, a_1, \dots$ are positive real numbers (a_0 may be zero)
- For $0 \le n \le m$, the nth *convergent* of the continued fraction $[a_0, \dots, a_m]$ is $c_n = [a_0, \dots, a_n]$. These convergents for $n < m$ are also called *partial convergents*.
- $[a_0, a_1, \dots, a_{n-1}, a_n] = \left[a_0, a_1, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right]$

Theorem

For each n with $-2 \leq n \leq m$, define real numbers p_n and q_n as follows:

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_0 &= a_0 \\ q_{-2} &= 1, & q_{-1} &= 0, & q_0 &= 1 \end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

Then, for $n \geq 0$ with $n \leq m$ we have

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$

(the last equality for $n \geq 1$)

Proof.

We use induction. The assertion is obvious when $n = 0, 1$. Suppose the proposition is true for all continued fractions of length $n - 1$. Then

$$\begin{aligned}
[a_0, \ldots, a_n] &= [a_0, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}] \\
&= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\
&= \frac{(a_{n-1}a_n + 1)p_{n-2} + a_n p_{n-3}}{(a_{n-1}a_n + 1)q_{n-2} + a_n q_{n-3}} = \frac{a_{n-1}a_n p_{n-2} p_{n-2} + a_n p_{n-3}}{a_{n-1}a_n q_{n-2} + q_{n-2} + a_n q_{n-3}} \\
&= \frac{a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \\
&= \frac{p_n}{q_n}.
\end{aligned}$$

□

Theorem

For $n \geq 0$ with $n \leq m$ we have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} \tag{1}$$

and

$$p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n. \tag{2}$$

Equivalently,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

and

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}}.$$

Proof.

The case for $n = 0$ is obvious from the definitions. Now suppose $n > 0$ and the statement is true for $n - 1$. Then

$$\begin{aligned} p_n q_{n-1} - q_n p_{n-1} &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - (a_n q_{n-1} + q_{n-2}) p_{n-1} \\ &= p_{n-2} q_{n-1} - q_{n-2} p_{n-1} \\ &= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\ &= -(-1)^{n-2} = (-1)^{n-1}. \end{aligned}$$

This completes the proof of (1). For (2), we have

$$\begin{aligned} p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2}) \\ &= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\ &= (-1)^n a_n. \end{aligned}$$

□

Theorem (How Convergents Converge)

The even indexed convergents c_{2n} increase strictly with n, and the odd indexed convergents c_{2n+1} decrease strictly with n. Also, the odd indexed convergents c_{2n+1} are greater than all of the even indexed convergents c_{2m}.

Proof.

The a_n are positive for $n \geq 1$, so the q_n are positive. By the previous theorem, for $n \geq 2$,

$$c_n - c_{n-2} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}},$$

which proves the first claim.

Suppose for the sake of contradiction that there exist integers r and m such that $c_{2m+1} < c_{2r}$. The previous theorem implies that for $n \geq 1$,

$$c_n - c_{n-1} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

has sign $(-1)^{n-1}$, so for all $s \geq 0$ we have $c_{2s+1} > c_{2s}$. Thus it is impossible that $r = m$. If $r < m$, then by what we proved in the first paragraph, $c_{2m+1} < c_{2r} < c_{2m}$, a contradiction (with $s = m$). If $r > m$, then $c_{2r+1} < c_{2m+1} < c_{2r}$, which is also a contradiction (with $s = r$). □

The continued fraction process I

- Let $x \in \mathbb{R}$ and write
$$x = a_0 + t_0$$
with $a_0 \in \mathbb{Z}$ and $0 \leq t_0 < 1$. We call the number a_0 the floor of x, and we also sometimes write $a_0 = \lfloor x \rfloor$.
- If $t_0 \neq 0$, write
$$\frac{1}{t_0} = a_1 + t_1$$
with $a_1 \in \mathbb{Z}$, $a_1 > 0$, and $0 \leq t_1 < 1$.
- Thus $t_0 = \frac{1}{a_1+t_1} = [0, a_1 + t_1]$, which is a continued fraction expansion of t_0, which need not be simple.

The continued fraction process II

- Continue in this manner so long as $t_n \neq 0$ writing

$$\frac{1}{t_n} = a_{n+1} + t_{n+1}$$

  with $a_{n+1} \in \mathbb{Z}$, $a_{n+1} > 0$, and $0 \leq t_{n+1} < 1$.
- We call this procedure, which associates to a real number x the sequence of integers $a_0, a_1, a_2, \ldots$, the *continued fraction process*.

Example

Let $x = \frac{1+\sqrt{5}}{2}$. Then

$$x = 1 + \frac{-1+\sqrt{5}}{2},$$

so $a_0 = 1$ and $t_0 = \frac{-1+\sqrt{5}}{2}$. We have

$$\frac{1}{t_0} = \frac{2}{-1+\sqrt{5}} = \frac{-2-2\sqrt{5}}{-4} = \frac{1+\sqrt{5}}{2},$$

so $a_1 = 1$ and $t_1 = \frac{-1+\sqrt{5}}{2}$. Likewise, $a_n = 1$ for all n. As we will see below, the following exciting equality makes sense.

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$$

Lemma

For every n such that a_n is defined, we have

$$x = [a_0, a_1, \dots, a_n + t_n],$$

and if $t_n \neq 0$, then $x = [a_0, a_1, \dots, a_n, \frac{1}{t_n}]$.

Proof.

We use induction. The statements are both true when $n = 0$. If the second statement is true for $n - 1$, then

$$\begin{aligned} x &= \left[a_0, a_1, \dots, a_{n-1}, \frac{1}{t_{n-1}}\right] \\ &= [a_0, a_1, \dots, a_{n-1}, a_n + t_n] \\ &= \left[a_0, a_1, \dots, a_{n-1}, a_n, \frac{1}{t_n}\right]. \end{aligned}$$

Similarly, the first statement is true for n if it is true for $n - 1$. □

Theorem

Let $a_0, a_1, \ldots$ be a sequence of **integers** *such that $a_n > 0$ for all $n \geq 1$, and for each $n \geq 0$, set $c_n = [a_0, a_1, \ldots a_n]$. Then $\lim_{n \to \infty} c_n$ exists.*

Proof.

For any $m \geq n$, the number c_n is a partial convergent of $[a_0, \ldots, a_m]$. We know that the even convergents c_{2n} form a strictly *increasing* sequence and that the odd convergents c_{2n+1} form a strictly *decreasing* sequence. Moreover, the even convergents are all $\leq c_1$ and the odd convergents are all $\geq c_0$. Hence $\alpha_0 = \lim_{n \to \infty} c_{2n}$ and $\alpha_1 = \lim_{n \to \infty} c_{2n+1}$ both exist, and $\alpha_0 \leq \alpha_1$. Finally,

$$|c_{2n} - c_{2n-1}| = \frac{1}{q_{2n} \cdot q_{2n-1}} \leq \frac{1}{2n(2n-1)} \to 0,$$

so $\alpha_0 = \alpha_1$. □

Theorem

Let $x \in \mathbb{R}$ be a real number. Then x is the value of the (possibly infinite) simple continued fraction $[a_0, a_1, a_2, \ldots]$ produced by the continued fraction procedure.

Proof

- If the sequence is finite, then some $t_n = 0$ and the result follows
- Suppose the sequence is infinite.
- Then

$$x = [a_0, a_1, \ldots, a_n, \frac{1}{t_n}].$$

- By a previous result (which we apply in a case when the partial quotients of the continued fraction are not integers), we have

$$x = \frac{\frac{1}{t_n} \cdot p_n + p_{n-1}}{\frac{1}{t_n} \cdot q_n + q_{n-1}}.$$

Proof (contd)

Thus, if $c_n = [a_0, a_1, \ldots, a_n]$, then

$$\begin{aligned}
x - c_n &= x - \frac{p_n}{q_n} \\
&= \frac{\frac{1}{t_n} p_n q_n + p_{n-1} q_n - \frac{1}{t_n} p_n q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)}. \\
&= \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)} \\
&= \frac{(-1)^n}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)}.
\end{aligned}$$

Proof (contd)

Thus

$$\begin{aligned}|x - c_n| &= \frac{1}{q_n\left(\frac{1}{t_n}q_n + q_{n-1}\right)} \\ &< \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} \\ &= \frac{1}{q_n \cdot q_{n+1}} \leq \frac{1}{n(n+1)} \to 0.\end{aligned}$$

In the inequality, we use that a_{n+1} is the integer part of $\frac{1}{t_n}$, and is hence $\leq \frac{1}{t_n}$.

Approximating reals with rationals

- $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- $\mathbb{Q}$ is dense in $\mathbb{R}$
- Thus, given $\alpha \in \mathbb{R}$ and $\mathbb{R} \ni \epsilon > 0$, exists $p, q \in \mathbb{Z}$ with $|\alpha - \frac{p}{q}| < \epsilon$
- However, $|q|$ may be very large
- Would like a compromise between good approximation and small denominator
- Continued fractions are very useful! They provide "the best" approximations, and can actually be used to show that certain numbers have few "good" approximations
- Applications:
  - Proving inequalities
  - Recognizing that α actually was a rational number
  - Proving that a real number is transcendental (Roth's theorem)

Obvious approximation

Lemma

For $\alpha \in \mathbb{R}$, $\mathbb{Z} \ni N > 1$, there are $p, q \in \mathbb{Z}$ with

$$|\alpha - \frac{p}{q}| < \frac{1}{N}, \qquad |q| \leq N$$

Proof.

WLOG $\alpha > 0$. Take $q = N$, $p = \lfloor \alpha N \rfloor$. □

Dirichlet's pigenhole

Theorem

For $\alpha \in \mathbb{R}$, $\mathbb{Z} \ni N > 1$, *there are* $p, q \in \mathbb{Z}$ *with*

$$|\alpha q - p| < \frac{1}{N}, \qquad |q| \leq N$$

Proof

- Consider the $N+1$ numbers

$$\{0\}, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\} \in [0, 1)$$

Proof (contd)

- Subdivide $[0, N)$ into N intervals

$$\left[0, \frac{1}{N}\right), \left[\frac{1}{N}, \frac{2}{N}\right), \ldots, \left[\frac{N-1}{N}, 1\right)$$

- At least one "collision": $i \neq j$, and m s.t.

$$\{i\alpha\}, \{j\alpha\} \in \left[\frac{m}{N}, \frac{m+1}{N}\right)$$

- $q = j - i$, $0 < q \leq N$
- $i\alpha = s + \{i\alpha\}$, $j\alpha = t + \{j\alpha\}$
- $(j - i)\alpha = t - s + \{j\alpha\} - \{i\alpha\}$
- $p = t - s$, then

$$|q\alpha - p| = |(j - i)\alpha - (t - s)| = |\{j\alpha\} - \{i\alpha\}| < \frac{1}{N}$$

Theorem (Kronecker)

Let α be irrational. For all real β and all $\epsilon > 0$, there exists integers A, c with

$$|a\alpha - \beta - c| < \epsilon$$

In other words, the sequence $(\{n\alpha\})_{n=1}^{\infty}$ is dense in $[0, 1)$.

Proof

- Dirichlet: exists integers a, b with $|a\alpha - b| < \epsilon$, $0 < a \leq \frac{1}{\epsilon}$
- α irrational, so $0 < |a\alpha - b|$
- Assume WLOG that $a\alpha - b > 0$, then

$$0 < a\alpha - b = \{a\alpha\} < \epsilon$$

Proof (contd)

- $$0, \{1a\alpha\}, \{2a\alpha\}, \dots \in [0, 1)$$
  and the distance between two closest points is $< \epsilon$
- So
  $$\{ka\alpha\} \leq \{\beta\} < \{(k+1)a\alpha\}$$
  some k, intervall length $< \epsilon$
- So
  $$0 < \{\beta\} - \{ka\alpha\} < \epsilon$$
- Hence exists integers A, c with
  $$|A\alpha - \beta - c| < \epsilon$$

From the proof of theorem on convergence of continued fractions we extract the following extremely useful tidbit:

Theorem (Convergence of continued fraction)

Let $a_0, a_1, \dots$ define a simple continued fraction, and let $x = [a_0, a_1, \dots] \in \mathbb{R}$ be its value. Then for all m,

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{q_m \cdot q_{m+1}} < \frac{1}{m^2}.$$

In other words,

$$|xq_m - p_m| < \frac{1}{q_{m+1}} < \frac{1}{m+1}.$$

Surprisingly, algebraic numbers (which should be more similar to rationals compared to transcendental numbers) can not be approximated better than that:

Theorem (Roth)

Let $\alpha \in \mathbb{R}$ and suppse there exists a positive real number ϵ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

for infinitely many different rationa numbers $\frac{p}{q}$. Then α is transcendental.

You can not find better rational approximations than the convergents stemming from continued fractions:

Theorem

Let $\alpha \in \mathbb{R}$, $s, r \in \mathbb{Z}$. Let, for positive k, $c_k = p_k/q_k$ be the k'th partial convergent of in the continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$.

- *If $|s\alpha - r| < |q_k\alpha - p_k|$ then $s \geq q_{k+1}$*
- *If $\left|\alpha - \frac{r}{s}\right| < \left|\alpha - \frac{p_k}{q_k}\right|$ then $s > q_k$*
- *If $\left|\alpha - \frac{r}{s}\right| - \frac{1}{2s^2}$ then $\frac{r}{s}$ is some partial convergent of α.*

Proof.

We show how the second part follows from the first. If $s \leq q_k$ and $|\alpha - r/s| < |\alpha - p_k/q_k|$ then, multiplying the inequalities, we get

$$s|\alpha - r/s| < q_k|\alpha - p_k/q_k|$$

hence

$$|s\alpha - r| < |q_k\alpha - p_k|,$$

a contradiction. □

- $0 < \alpha \in \mathbb{R}$
- Study the line $y = \alpha x$ with slope α
- Lattice points $(q, p) \in \mathbb{Z}^2$ that are close to the line correspond to rational approximations p/q that are close to α
- If $\alpha \notin \mathbb{Q}$ then the line splits lattice points in pos quadrant in two parts, one above and one below
- Vertices of the convex hulls of these subsets correspond to best approximations, i.e. to partial convergents
- The recursions

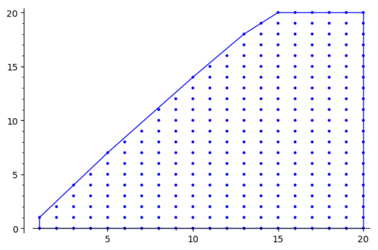
$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

can be interpreted as a recursion for directional vectors:

$$\begin{pmatrix} q_n \\ p_n \end{pmatrix} = a_n \begin{pmatrix} q_{n-1} \\ p_{n-1} \end{pmatrix} + \begin{pmatrix} q_{n-2} \\ p_{n-2} \end{pmatrix}$$

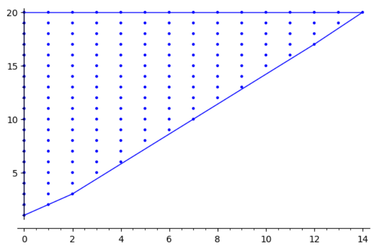
Example

Let $\alpha = \sqrt{2}$. Rather than looking at the infinitely many lattice points in the positive quadrant, we split $[0, 20]^2$ into two parts, onew with $y > \sqrt{2}x$, one with $y < \sqrt{2}x$.



Relevant vertices of convex hull: $(1, 1), (5, 7)$

Example

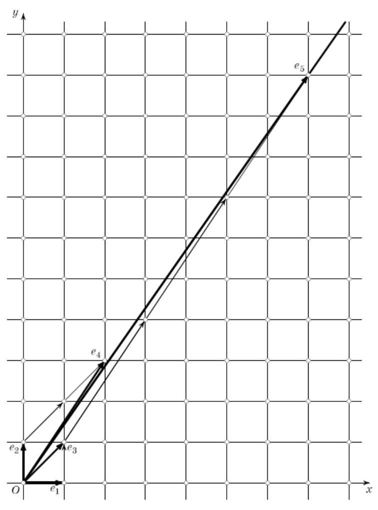


Relevant vertices of convex hull: $(2,3), (12,17)$

Convergents of $\sqrt{2} = [1,2,2,2,2,\ldots]$ is 1, 3/2, 7/5, 17/12, $\ldots$.

Example

$$\frac{10}{7} = 1 + \frac{1}{2+\frac{1}{3}}$$



Approximating real numbers, recognizing rational numbers

- General principle: if a_n large, then $|[a_0, \ldots, a_{n-1}] - [a_0, \ldots, a_{n-1}, a_n]|$ small
- Cutting of at this point makes an unusually good approximation

Example

- $\alpha = \frac{1234}{6789} \approx 0.181764619237001\ 031079687730151716\ 0\ 111945794667845043\ 45\ 264398291353660332\ 89\ 144203859183974075\ 71\ 070849904256886139\ 34\ 3054941817646192$
- Since $\gcd(1234, 6789) = 1$ and $\gcd(2, 6789) = 1$ and $\gcd(5, 6789) = 1$ and $\text{ord}_{6789}(10) = 120$, thm 12.4 in Rosen gives that the decimal expansion of α is periodic, right off the bat, with period 120. You can see 181764etc occuring right at the end
- Given just the decimal expansion of α, you thus need at least 120 digits to recognize it as a rational

Example

- $\alpha = [0; 5, 1, 1, 153, 1, 3]$.
- We calculate the CF of truncations of the decimal expansion of α.
- We swiftly get the correct answer by truncating before huge entry; recall that $[a_0, \ldots, m] = [a_0, \ldots, m-1, 1]$

k	$n(\alpha, k)$	CF
1	0.1	[0; 10]
2	0.18	[0; 5, 1, 1, 4]
3	0.181	[0; 5, 1, 1, 9, 1, 1, 4]
4	0.1817	[0; 5, 1, 1, 69, 2, 1, 1, 2]
5	0.18176	[0; 5, 1, 1, 141, 2]
6	0.181764	[0; 5, 1, 1, 151, 1, 73, 2]
7	0.1817646	[0; 5, 1, 1, 153, 1, 2, 3, 1, 1, 1, 4, 5, 1, 2]
8	0.18176461	[0; 5, 1, 1, 153, 1, 2, 1, 1, 1, 1, 2, 54, 7, 1, 2]
9	0.181764619	[0; 5, 1, 1, 153, 1, 2, 1, 90, 1, 3, 1, 9, 1, 1, 2, 1, 1, 2]
10	0.1817646192	[0; 5, 1, 1, 153, 1, 2, 1, 585, 1, 1, 1, 1, 1, 19]
11	0.18176461923	[0; 5, 1, 1, 153, 1, 2, 1, 3098, 3, 1, 1, 18, 1, 4, 1, 5]

Number of days in a year

Example

The number of days (rotations around the earth's axis) in a year (one revolution around the sun) is approximately

$$\alpha = 365.2422 = [365, 4, 7, 1, 3, 4, 1, 1, 2]$$

The second convergent is

$$\alpha \approx 365.25 = 365 + \frac{1}{4}$$

which gives a calendar with a leap year every fourth year. The fourth convergent gives

$$\alpha \approx [365, 4, 7, 1] = 365 + \frac{8}{33} \approx 365.2424\ldots$$

Example (contd)

The Gregorian calendar has

- A leap year every fourth year
- Except, if the year is divisibly by 100, no leap year
- Except, if actually divisible by 400, a leap year

Thus the number of days per year is

$$365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} = 146097/400 = [365, 4, 8, 12] = 365.2425$$

which is very close to $365 + \frac{8}{33} \approx 365.2424$

Example (Lunar calendar)

The number of lunations (revolutions of the moon around the earth) in one year is approximately

$$\beta \approx 12.368267 \approx [12; 2, 1, 2, 1, 1, 17, 2, 2, 15]$$

We use the fifth convergent,

$$[12; 2, 1, 2, 1, 1] = 235/19$$

to explain the 19-year Metonic cycle in the hebrew calendar:

$$235 = 19 * 12 + 7$$

and the Hebrew calendar inserts 7 extra months in one 19-year Metonic cycle.

Huygen's planetarium

Huygen's mechanical model of the solar system used gear ratios to produce properly scaled planetary orbits. For example, since the time it takes for Saturn to make one revolution around the sun is, measured in earth years, approximately

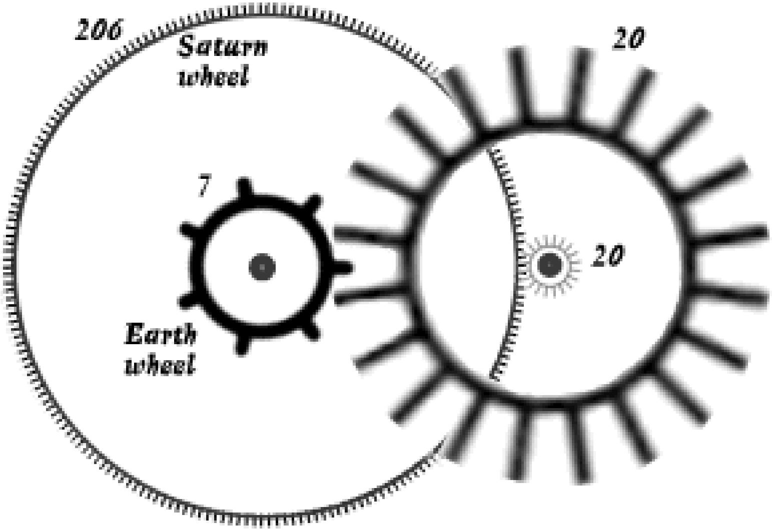
$$\frac{77708431}{2640858} = 29.425448\cdots = [29, 2, 2, 1, 5, 1, 4, \ldots]$$

we can use the fourth convergent

$$[29, 2, 2, 1] = \frac{206}{7}$$

to make a gear with 7 teeth for earth, and a gear with 206 teeth for Saturn.

Huygen's planetarium



Example

The golden ratio Φ is the positive root of the polynomial

$$x^2 = x + 1,$$

so

$$x = 1 + \frac{1}{x}.$$

Iterating, we get

$$x = 1 + \frac{1}{x} = 1 + \frac{1}{1 + \frac{1}{x}} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{x}}} = \cdots = [1, 1, 1, \ldots]$$

Definition

- An infinite CF α is periodic with (least) period k and preperiod N if

$$\alpha = [a_0, a_1, a_2, \ldots]$$

  with

  - $a_{m+k} = a_m$ for all $m \geq N$
  - k is smallest with this property
  - N is smallest with this property

- We write

$$\alpha = [a_0, \ldots, a_{N-1}, \overline{a_N, \ldots, a_{N+k-1}}]$$

- If the preperiod is 0 the α has a purely periodic CF expansion

Example

- Let $\alpha_0 = \alpha = \sqrt{2}$
- $a_0 = \lceil \alpha_0 \rceil = 1,\ t_0 = \alpha_0 - a_0 = \sqrt{2} - 1$
- $\alpha_1 = \frac{1}{t_0} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2} + 1$
- $a_1 = \lceil \alpha_1 \rceil = 2,\ t_1 = \alpha_1 - 2 = \sqrt{2} - 1 = t_0$
- $\alpha_2 = \alpha_1$, but we always have $\alpha_1 = a_1 + \frac{1}{\alpha_2}$
- So α_1 is a solution to $t = 2 + \frac{1}{t}$, or $t^2 - 2t - 1 = 0$, and $\alpha_1 = [2, 2, 2, \ldots] = [\overline{2}]$
- Finally, $\alpha = \alpha_1 - 1 = [1, 2, 2, \ldots] = [1, \overline{2}]$

Example

Suppose that

$$\beta = [3, 5, \overline{7, 11}]$$
$$\gamma = [\overline{7, 11}]$$

Then

$$\gamma = 7 + \frac{1}{11 + 1/\gamma} = 7 + \frac{\gamma}{11\gamma + 1} = \frac{78\gamma + 7}{11\gamma + 1}$$

so γ is a root of

$$11t^2 - 77t - 7,$$

in fact, $\gamma = \frac{9}{22}\sqrt{77} + \frac{7}{2}$ Furthermore,

$$\beta = 3 + \frac{1}{5 + \frac{1}{\gamma}} = \frac{9}{442}\sqrt{77} + \frac{1333}{442}$$

Definition

- Let $\alpha \in \mathbb{R}$ be algebraic over $\mathbb{Q}$, with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$, id est, $\alpha \notin \mathbb{Q}$ and exists $A, B, C \in \mathbb{Z}$ such α is a zero of the polynomial

$$At^2 + Bt + C = 0$$

  Then α is called a quadratic irrationality.

- The other zero of this polynomial is called the algebraic conjugate of α, and is denoted α'. Thus

$$At^2 + Bt + C = A(t - \alpha)(t - \alpha')$$

- α is reduced if $\alpha > 1$ and $-1 < \alpha' < 0$

Lemma

Let α be the the previous quadratic irrationality

- *If r, s, t, u are integers, then*

$$\frac{r\alpha + s}{t\alpha + u} \in \mathbb{Q}(\alpha)$$

  so it is rational or a quadratic irrationality
- $\alpha = (a + \sqrt{b})/c$ *with* a, b, c *integers,* $b > 0$, b *not a perfect square*
- *Then* $\alpha' = (a - \sqrt{b})/c$
- *This conjugation extends to a field automorphism on* $\mathbb{Q}(\alpha)$ *by*

$$(r\alpha + s)' = r\alpha' + s$$

Lemma

If α is a quadratic irrationality, then

$$\alpha = \frac{P + \sqrt{d}}{Q}$$

with P, Q, d integers, $d > 0$, d not a perfect square, $Q | (d - P^2)$

Proof.

$$\alpha = \frac{a + \sqrt{b}}{c} = \frac{|c|a + |c|\sqrt{b}}{|c|c} = \frac{|c|a + \sqrt{bc^2}}{|c|c} = \frac{P + \sqrt{d}}{Q}$$

□

Theorem (Lagrange)

The continued fraction expansion of a real number x is

- *finite if and only if x is rational*
- *ultimately periodic if and only if x is a quadratic irrationality*
- *purely periodic if and only if x is a reduced quadratic irrationality*

Example

- $\frac{173}{37} = [4, 1, 2, 12]$
- $\frac{5+\sqrt{3}}{2} = [3, \overline{2, 1}]$
- $\frac{1+\sqrt{3}}{2} = [\overline{1, 2}]$

Theorem

Let α be a quadratic irrationality. The the following algorithm calculates the continued fraction expansion of α:

1. *INITIALIZE:*
   - $\alpha_0 = \alpha$
   - $\alpha_0 = \frac{P_0+\sqrt{d}}{Q_0}$
   - $k = 0$
2. *UPDATE:*
   - $a_k = \lfloor \alpha_k \rfloor$
   - $P_{k+1} = a_k Q_k - P_k$
   - $Q_{k+1} = (d - P_{k+1}^2)/Q_k$

Then for there exists some n so that for $k > n$, $0 < Q_k \leq d$, $-\sqrt{d} < P_k < \sqrt{d}$, so eventually there is some N, ℓ such that $P_N = P_{N+\ell}$, $Q_N = Q_{N+\ell}$, and the algorithm starts repeating itself cyclically.

Example

- $\alpha = \alpha_0 = \sqrt{5} = \frac{0+\sqrt{5}}{1}$
- $a_0 = 2,\ P_0 = 0, Q_1 = 1$
- $P_1 = a_0 Q_0 - P_0 = 2,\ Q_1 = (5 - P_1^2)/Q_0 = 1$
- $\alpha_1 = \frac{2+\sqrt{5}}{1},\ a_1 = 4$
- $P_2 = a_1 Q_1 - P_1 = 2,\ Q_2 = (5 - P_2^2)/Q_1 = 1$
- Repeating, so $\sqrt{5} = [2, \overline{4}]$

What about algebraic numbers of higher degree?

- - Quadratic number
  - ultimately periodic CF $[a_0, a_1, a_2, \ldots]$
  - 1-automatic, i.e. produced by finite state automaton with input alphabet of size one
- Higher degree: CF not automatic sequence
- Khintchine: for "almost all" reals the geometric mean $\left(\prod_{j=1}^{n} a_j\right)^{1/n}$ tends to my constant!
- Though the CF of algebraic numbers of higher degree lack structure, thay can still be computed efficiently

Example

- $f_0(t) = t^3 - 2$, $\alpha_0 = 2^{1/3} = [1; 3, 1, 5, 1, 1, 4, \ldots]$, but we don't know that
- We do know that $1 < \alpha_0 < 2$, since $f_0(1) < 0$ and $f_0(2) > 0$
- So $a_0 = 1$, (and $\alpha_1 = [3, 1, 5, 1, 1, 4, \ldots]$)
- $f_1(t) = -t^3 f_0(a_0 + 1/t) = t^3 - 3t^2 - 3t - 1$
- $f_1(3) < 0$, $f_1(4) > 0$, so $a_1 = 3$
- $f_2(t) = -t^3 f_1(a_1 + 1/t) = 10t^3 - 6t^2 - 6t - 1$
- $f_2(1) < 0$, $f_2(2) > 0$, so $a_2 = 1$
- Et cetera
- One can show that each $f_k(t)$ has a single zero, which is α_k, and which is located between a_k and $a_k + 1$, and that $f(a_k) < 0$, $f(a_k + 1) > 0$